Stability of Superconducting States out of Thermal Equilibrium

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Quasiparticles of superconductors can be manipulated either by irradiation or tunnel injection such that substantial changes of the gap (order parameter) occur which are most pronounced close to the transition temperature. These phenomena are investigated by a theory which is based on the combined system of the Boltzmann equation and the BCS gap equation. Several stationary, spatially homogeneous states are found in some range of temperatures. A linear stability analysis reveals two types of behavior, depending, essentially, on whether quasiparticle diffusion increases or decreases the stability. In a representative situation of the first type, two states turn out to be locally stable. An investigation of a nonlinear equation of motion for the order parameter leads to the conclusion that a first-order phase transition between these two states occurs at a given temperature. In a situation of the second type, one finds that fluctuations of a definite wave vector destroy the spatially homogeneous state and lead to a stable state of layered structure.

1. INTRODUCTION

Superconducting states out of thermal equilibrium have been investigated recently, experimentally as well as theoretically.* They may exist when the metal is exposed to (steady) external perturbations such as irradiation by lasers, with microwaves, etc., or by the tunnel injection of quasiparticles. Sometimes superconductivity has been found to be enhanced, and to exist even above the thermodynamic transition temperature. The question of the stability of these states has been raised, in particular, by the observation of transitions to the normal state, which in some cases appear to occur gradually and in other cases abruptly. The experimental evidence leaves this question open, and theoretical models have been proposed to explain the observations.

*For general information, we refer to the reviews cited in Refs. 1 and 2.

Clearly, the systems we are considering are open systems, which means, for instance, that energy is continuously entering and leaving, in different forms. Open systems are, in principle, different from the closed systems one considers in thermodynamics. In order to make the difference clear, we will call the states of the open system dissipative states, in contrast to the thermal states which are the steady states of a closed system.*

Steady states may also exist in an open system. Obviously, such states are characterized by a dynamic equilibrium among the various competing processes.† Quite generally, a theory of steady dissipative states and their stability will lack either the simplicity or the generality of the stability theory of thermal states. While a thermal equilibrium state of a closed system is characterized by equal probability of the microstates and by the principle of detailed balance, both characteristics are absent in dissipative states.

In the following, we will investigate dissipative superconducting states where the external perturbation changes the occupation number n_p of those quasiparticle states that are capable of blocking the phase space of the Cooper pairs. Considering the BCS gap equation[4]

$$\Delta = \lambda \int d\varepsilon_p \, (\Delta/2E_p)(1-2n_p) \tag{1}$$

where Δ is the energy gap or order parameter, λ the dimensionless coupling constant, and where $E_p = (\varepsilon_p^2 + \Delta^2)^{1/2}$, one recognizes that quasiparticles of low energy, typically in the range of some Δ's, play the most important role in blocking the pairs. Hence, we will mainly be concerned with just this type of quasiparticle.

Mathematically, the occupation number n_p (distribution function) is found as the solution of a Boltzmann equation.[5,6] This equation includes the well-known terms describing processes that tend to restore equilibrium, such as the drift and the collision of quasiparticles; but it also includes a drive term $(\dot{n}_p)_D$, which describes the change in quasiparticle occupation number due to the action of the perturbation. The basic components of the Boltzmann equation are displayed in Section 2, as is the Ginzburg–Landau form of the gap equation, which is most convenient, since the following investigations are concerned only with the case $\Delta \ll k_B T_c$.

In the next four sections, we investigate the case where the drive term is of the following form:

$$(\dot{n}_p)_D = -\text{const} \times \Delta^2/E_p \tag{2}$$

As shown by Elesin,[7] such a drive term arises in effect when a superconductor near T_c is exposed to optical radiation. In Section 3, we investigate the

*In this context, we draw attention to the review article by Haken.[3]

†Frequently, dissipative states are called "nonequilibrium states." We avoid this term, since, as discussed above, an equilibrium may exist.

local stability, i.e., stability against small fluctuations (linear stability analysis), of the stationary, spatially homogeneous states of the Elesin model system (EMS). We consider in Section 4 the possibility of first-order phase transitions between locally stable states. According to the theory, these phase transitions are driven by quantum fluctuations of the occupation numbers. Furthermore, we calculate the lifetime of the metastable state. The case where the quasiparticle diffusion length dominates the spatial changes of the order parameter is discussed in Section 5. In this case, detailed balance is violated macroscopically, and one cannot accomplish the extensive program of the preceding section. However, it is still possible to find the condition for the coexistence of the metastable phases. Section 6 contains an investigation of the Elesin model system with a negative coefficient (NEMS). This case is interesting because, there, quasiparticle diffusion induces instability, which eventually leads to a spatially modulated stable state.

Further examples of dissipative states are studied in the next three sections. We consider microwave-stimulated superconductivity in Section 7. In this case, the drive term is rather complicated, but one can show that it leads to qualitatively the same results as EMS. In Section 8, we investigate phonon-stimulated superconductivity, which leads to a variety of dissipative states. For large values of the phonon radiation power, and for small values of the gap, we find spatially modulated states to be stable, as in NEMS. Section 9 contains a discussion of stimulation by quasiparticle tunneling, where various dissipative states may occur. The case is most interesting since, inherently, the experiment contains a control mechanism which stabilizes the phase coexistence.

Finally, we give a general discussion in Section 10 of our results and of related theoretical and experimental work.

2. MICROSCOPIC THEORY

Our basic understanding, supported by experience, is that the BCS theory also applies to superconductors away from thermal equilibrium. This means that the superconducting states can be described by local values (i.e., local in space and time) of the gap and of the quasiparticle distribution function. For reasons of transparency, we formulate this theory in the limit of weak pair-breaking,* where the quasiparticles have definite momentum and energy. When necessary we generalize the final result, as, for instance, by the inclusion of the order parameter deformation energy known from the Ginzburg-Landau theory. We investigate only those states where the order

*In the general case, one has to use the formulation of Refs. 8–10.

parameter Δ is real,* and where the quasiparticle distribution function n_p is an even function of ε_p. The Boltzmann equation is of the form[6]

$$\dot{n}_p + [E_p, n_p]_{\mathrm{PB}} + I\{n_p\} = (\dot{n}_p)_D \tag{3}$$

where

$$[E_p, n_p]_{\mathrm{PB}} = \frac{\partial E_p}{\partial p}\frac{\partial n_p}{\partial r} - \frac{\partial E_p}{\partial r}\frac{\partial n_p}{\partial p} \tag{4}$$

denotes the Poisson bracket.† We recognize that the distribution function may change in time by convection, by collisions, and by external perturbations. The convection is represented in the canonical form of a Poisson bracket. The collisions are represented by a collision integral which, in the case of a superconductor, contains only two important contributions. These arise from collisions of the quasiparticles with impurities and with phonons; hence

$$I\{n_p\} = I_{\mathrm{imp}}\{n_p\} + I_{ep}\{n_p\} \tag{5}$$

Impurity scattering is elastic, and if, in addition, this scattering is isotropic, we obtain the simple expression

$$I_{\mathrm{imp}}\{n_p\} = \frac{1}{\tau_i}\frac{|\varepsilon_p|}{E_p}(n_p - \langle n_p\rangle) \tag{6}$$

where $\langle\cdots\rangle$ denotes the angular average. As far as the collisions with phonons are concerned, we will assume throughout that the phonons are in a state of internal equilibrium which can be characterized by a temperature T close to T_c and that the energy gap Δ is much smaller than $k_{\mathrm{B}}T_c$. In such a collision, a quasiparticle changes its energy by an amount of the order of $k_{\mathrm{B}}T$. Since the quasiparticles that are of interest in the gap equation have energies of the order of Δ ($\ll k_{\mathrm{B}}T_c$), we may approximate‡ I_{ep} by the relaxation ansatz

$$I_{ep}\{n_p\} = (1/\tau_E)(n_p - n_T) \tag{7}$$

where $n_T = n_T(E_p)$ is the thermal distribution (Fermi function). As E and Δ are small, we consider the inelastic collision rate $1/\tau_E$ to be constant, equal to its value at $E = 0$ and $T = T_c$.

*In the terminology of Ref. 9, these states have to be identified with the L-mode. Properly speaking, there, the *isotropic* part of n_p is an even function of ε_p.

†Since at no important step is it likely that a misunderstanding will occur, we do not denote vector quantities by a special symbol.

‡The exact expression is given in Appendix A.

The assumptions made above allows us to write the gap equation in the Ginzburg–Landau form

$$-[\alpha+\beta(\Delta^2/T_c^2)-\chi]\Delta=0 \tag{8}$$

where $\alpha=(T-T_c)/T_c$ and $\beta=7\zeta(3)/8\pi^2$.* Furthermore, we have introduced the quantity

$$\chi=-\int d\varepsilon_p\,(1/E_p)\langle n_p-n_T\rangle \tag{9}$$

Since the gap is controlled by the temperature and by this quantity, we will call χ the control function or the gap control. Of course, the Boltzmann equation connects the control function with the gap—ultimately it is a problem of self-consistency.

3. LOCAL STABILITY OF EMS

As shown in detail in Appendix A, the drive term introduced in Eq. (2) has its origin in the peculiar form of the BCS coherence factors for quasiparticle–phonon collisions, which leads to an increase of the quasiparticle recombination rate with increasing energy gap. Hence, it is essentially negative, and we may write

$$(\dot{n}_p)_D=-\frac{B}{\pi\tau_E T_c}\frac{\Delta^2}{E_p} \tag{10}$$

where B is a dimensionless quantity proportional to the radiation power.

Considering Eqs. (3) and (7), we find that in a steady, spatially homogeneous state

$$n_p-n_T=-\frac{B}{\pi T_c}\frac{\Delta^2}{E_p} \tag{11}$$

Inserting this result in Eq. (9), we obtain for the gap control

$$\chi=\int d\varepsilon\,\frac{B\Delta^2}{\pi T_c E^2}=B\frac{\Delta}{T_c} \tag{12}$$

Solving Eq. (8), one finds that χ enhances the gap, as shown in Fig. 1. Most interesting is the fact that, for a given temperature in the range $T_c<T<T_M$, there exist two superconducting states with different values of Δ, which we will refer to as the states on the lower and the upper branch. In particular, we have

$$\frac{\Delta}{T_c}=\frac{B\pm(B^2-4\beta\alpha)^{1/2}}{2};\qquad \alpha_M=\frac{B^2}{4\beta} \tag{13}$$

*In the following, we use units where $\hbar=k_B=1$.

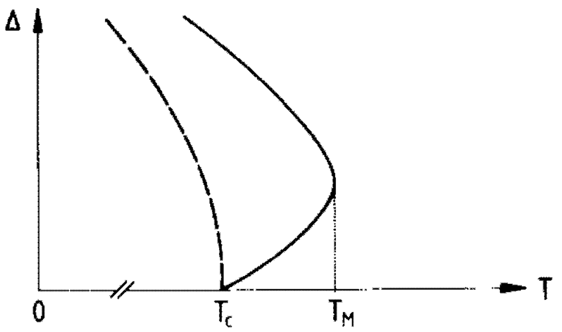


Fig. 1. Temperature dependence of the energy gap (near T_c) in a stimulated superconductor (dashed line: thermal equilibrium).

where $\alpha_M = (T_M - T_c)/T_c$. We note in passing that Eq. (8) always allows the normal state solution $\Delta = 0$.

A state of the system may be looked at as a point in the phase space spanned by the relevant variables. A small fluctuation means a "local" excursion of this point to the vicinity of the reference state, which may be any one of the stationary states found above. A state is called locally stable (unstable) if any (at least one) such fluctuation decays (grows) in time. We consider first spatially homogeneous fluctuations where the relevant quantities acquire increments

$$\Delta \to \Delta + \delta\Delta, \qquad n_p \to n_p + \delta n_p \tag{14}$$

dependent only on time, such that the time variation is governed by a single frequency ω. Hence, the time derivative in the Boltzmann equation takes the form*

$$\dot{n}_p = (-i\omega)\left(n'_T \frac{\Delta\,\delta\Delta}{E_p} + \delta n_p\right) \tag{15}$$

where n'_T is the energy derivative of the Fermi function. Obviously, δn_p is isotropic in p, and we obtain

$$\delta n_p = \frac{1}{1 - i\omega\tau_E}\,\tau_E\left[\delta(\dot{n}_p)_D + i\omega n'_T \frac{\Delta\,\delta\Delta}{E_p}\right] \tag{16}$$

where $\tau_E\,\delta(\dot{n}_p)_D = -(B/\pi T_c)\,\delta(\Delta^2/E_p)$. Hence, the control function changes by

$$\delta\chi = -\int d\varepsilon_p \frac{1}{E_p}\,\delta n_p - \int d\varepsilon_p\left(\delta\frac{1}{E_p}\right)\langle n_p - n_T\rangle \tag{17}$$

*The contribution from $(n_p - n_T)$ to the time derivative is negligible since $B \ll 1$.

and there is a corresponding change in the Ginzburg–Landau equation

$$-\delta\Delta\frac{\partial}{\partial\Delta}\left(\alpha+\beta\frac{\Delta^2}{T_c^2}-B\frac{\Delta}{T_c}\right)\Delta+\frac{i\omega\tau_E\,\delta\Delta}{1-i\omega\tau_E}\frac{\pi\Delta}{4T_c}[1+O(B)]=0 \tag{18}$$

which is a sum of a static and of a dynamic part.

Considering the states on the upper branch of Fig. 1, one finds that the static part is negative. As a consequence, there is no exponentially growing ($\mathrm{Im}\,\omega>0$) solution of Eq. (18) and hence these states are stable. On the other hand, an exponentially growing solution exists for states on the lower branch, which shows that they are unstable.

We note in passing that there exist exponentially decaying ($\mathrm{Im}\,\omega<0$) solutions for the stable states on the upper branch. This allows us to define a characteristic time $\tau_{R\Delta}$ for gap relaxation which is of the order $(T_c/\Delta)\tau_E \gg \tau_E$. This means that in most cases we may neglect the term $\omega\tau_E$ in the denominator of the dynamic part of Eq. (18). Then, one can also derive a nonlinear time-dependent Ginzburg–Landau equation

$$\frac{\pi\tau_E\Delta}{4T_c}\dot{\Delta}=-\left(\alpha+\beta\frac{\Delta^2}{T_c^2}-B\frac{\Delta}{T_c}\right)\Delta \tag{19}$$

of which Eq. (18) is the linearized form.

For the discussion of the normal state stability, one has to add to the left side of Eq. (19) a term $(\pi/8T_c)\,\partial\Delta/\partial t$, which one obtains when using the more general Green's function technique. Thus, one may prove that the normal state is stable above $T_c(\alpha>0)$ and unstable below $T_c(\alpha<0)$.

In the next step, we consider spatially inhomogeneous fluctuations, which shall be characterized by a single wave vector q. In such a case, it is advantageous to put the Boltzmann equation in a different form and to rewrite the Poisson bracket in the form of a Jacobian

$$[E_p,n_p]_{\mathrm{PB}}=\frac{\partial(E,n)}{\partial(p,r)}=\frac{\partial(n,E)}{\partial(r,E)}\cdot\frac{\partial(r,E)}{\partial(r,p)}=\left(\frac{\partial E}{\partial p}\right)_r\cdot\left(\frac{\partial n}{\partial r}\right)_E \tag{20}$$

This relation shows us that it is convenient to use a representation where the energy, and not the momentum, labels the distribution function.* Therefore, if $\eta_p=n_p-n_T(E_p)$ is any deviation from local equilibrium, we put $\eta_p=\eta(E,\hat{p})$, where $\hat{p}$ is a vector parallel to p and of length p_F. We emphasize that in the following we will perform all derivatives at constant

*We may illustrate this as follows. Consider the case where the gap varies in space. It is clear that the quasiparticles drift through space at constant energy E and not at constant ε_p. On the other hand, if the gap varies in time, the quasiparticles remain in the same momentum state. Hence, the drift in time is characterized by $\varepsilon_p=$ const.

energy. Since $[E_p, n_T]_{PB} = 0$, the Boltzmann equation for η takes the form

$$\frac{\varepsilon}{E}\frac{\hat{p}}{m}\nabla\eta(E,\hat{p}) + I\{\eta\} = (\dot{n}_E)_D - n'_T\frac{\Delta\dot{\Delta}}{E} \tag{21}$$

where $\dot{\eta}$ has been omitted since $\omega\tau_E \ll 1$. Furthermore, $\varepsilon = \pm(E^2-\Delta^2)^{1/2}$, and ∇ and ∂/t correspond to iq and $-i\omega$, respectively. Considering the form (6) for I_{imp}, we realize that in the case $ql \ll 1$, the anisotropic part $\eta^{(a)}$ of the distribution function is connected with the isotropic part $\eta(E) = \langle\eta(E,\hat{p})\rangle$ by the following relation:

$$\eta^{(a)}(E,\hat{p}) = -(\operatorname{sgn}\varepsilon)\tau_i(\hat{p}/m)\nabla\eta(E) \tag{22}$$

Inserting this form in Eq. (21), and averaging over all directions, we obtain the following diffusive Boltzmann equation*:

$$-\frac{|\varepsilon|}{E}D\,\nabla^2\eta(E) + I_{\text{ep}}\{\eta\} = (\dot{n}_E)_D - n'_T\frac{\Delta\dot{\Delta}}{E} \tag{23}$$

where $D = (1/3)v_F^2\tau_i$ is the diffusion constant.

In particular, Eq. (23) applies to the local fluctuation $\delta n_p = \delta n(E,\hat{p})$ [respectively, $\delta n(E) = \langle\delta n(E,\hat{p})\rangle$] if the fluctuating quantities are inserted on the right-hand side. Neglecting for the moment the term with the time derivative $\delta\dot{\Delta}$, we obtain immediately

$$\delta n(E) = \frac{1}{1+(|\varepsilon|/E)\Lambda^2q^2}\,\tau_E\,\delta(\dot{n}_E)_D \tag{24}$$

where $\Lambda = (D\tau_E)^{1/2}$ is the inelastic diffusion length. It is clear that now $\tau_E\,\delta(\dot{n}_E)_D$ is meant to be equal to $-2(B/\pi)(\Delta\,\delta\Delta/T_cE)$. In the energy representation, the change in the gap control is given by

$$\delta\chi_q = -\int dE\,N_1(E)\frac{1}{E}\,\delta n(E) - \int dE\,[\delta N_1(E)]\frac{1}{E}\,\tau_E(\dot{n}_E)_D \tag{25}$$

where $N_1(E) = \theta(E^2-\Delta^2)|E|(E^2-\Delta^2)^{-1/2}$ is the reduced BCS density of states.† Some care has to be exercised in the evaluation of this expression since the variation of the density of states with respect to the gap is highly singular. In the present case, we obtain

$$\delta\chi_q = \frac{B\,\delta\Delta}{T_c}[2h(\Lambda^2q^2)-1] \tag{26}$$

The function $h(\Lambda^2q^2)$ results from the first integration of Eq. (25); it is given

*We remark that in the Green's function technique, one arrives immediately at the energy representation of the Boltzmann equation. See, for instance, Ref. 9. Actually, the condition $\omega \ll \text{Max}\,(1/\tau_E, Dq^2)$ has to be satisfied for the validity of Eqs. (21) and (23), respectively.

†For convenience, we include in the above integration positive and negative values of E.

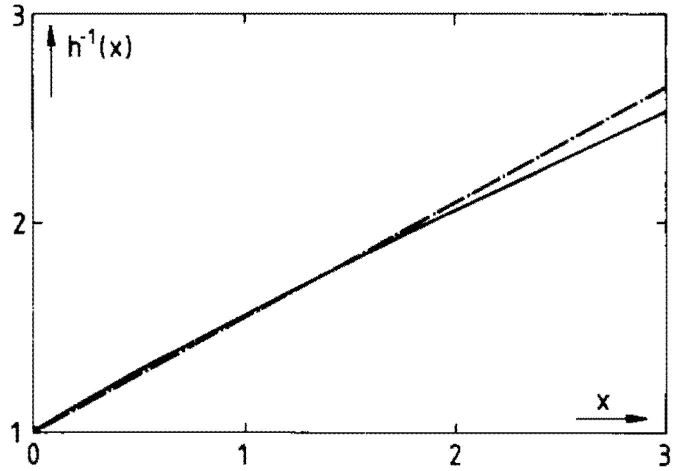


Fig. 2. Plot of $h^{-1}(x)$ as given by Eq. (27) (solid line) and of $1+0.53x$ (broken line) vs. x.

by

$$h(x)=\frac{2}{\pi}\int_1^{\infty} dy \frac{1}{(y^2-1)^{1/2}}\frac{1}{y+x(y^2-1)^{1/2}}$$
$$=\frac{4}{\pi}\frac{1}{(x^2-1)^{1/2}}\ln\frac{(x+1)^{1/2}+(x-1)^{1/2}}{\sqrt{2}} \tag{27}$$

which decreases from 1 to 0 as x increases from 0 to ∞. A graph of $h^{-1}(x)$ vs. x is shown in Fig. 2.

We include now the term with the time derivative and obtain by a similar calculation

$$\delta\chi_\omega=(\pi i\omega\tau_E/4T_c)h(\Lambda^2q^2)\,\delta\Delta \tag{28}$$

From these results it follows that the gap equation is equivalent to a linearized Ginzburg–Landau equation of the form

$$-(\pi\Delta/4T_c)i\omega\tau_E h(\Lambda^2q^2)\,\delta\Delta=-X\,\delta\Delta \tag{29}$$

with

$$X=\frac{\partial}{\partial\Delta}\left(\alpha+\beta\frac{\Delta^2}{T_c^2}-B\frac{\Delta}{T_c}\right)\Delta+2B\frac{\Delta}{T_c}[1-h(\Lambda^2q^2)]$$

Inspecting the properties of X, one finds that it is positive for the states on the upper branch of Fig. 1, and that it increases with increasing q. This means that fluctuations decay in time faster if q increases.* As far as the unstable states on the lower branch are concerned, we may say that fluctuations increase slower in time as q increases, and eventually, they will even decay in

*Actually, they decay even faster, if we take the factor $h(\Lambda^2q^2)$ on the left side into account.

time for sufficiently large values of q. This behavior demonstrates clearly, for the dissipative states of EMS, that quasiparticle diffusion increases the stiffness of the states against fluctuations of increasing wave number.

In a last step, we include the Ginzburg–Landau deformation energy $\xi^2(0)q^2\,\delta\Delta$ in the definition of X by Eq. (29). Generally, this term has the effect of increasing the stiffness with increasing q.

We may summarize the results obtained in this section as follows. For $T < T_c$, only the superconducting state is stable. In the temperature range $T_c < T < T_M$, we have found three states where the state with an intermediate value of the gap is unstable.

4. DISSIPATIVE PHASE TRANSITIONS OF EMS

We examine now a nonlinear equation of motion for the order parameter in which we include as an essential element the intrinsic processes generating large-scale fluctuations. Thus, we arrive at a stochastic description of the order parameter motion. Then, it follows that only one of the two locally stable states can be stable in the sense of what we call global stability. In order to reach definite conclusions, we neglect, in this section, quasiparticle diffusion, which is possible if the Ginzburg–Landau coherence length is much larger than the quasiparticle diffusion length, i.e. $\xi(T) \gg \Lambda$. We will find that, as a function of temperature, this global stability shifts discontinuously from one state to the other, which indicates a first-order phase transition between the two locally stable states.

We arrive at the stochastic description by taking into account the quantum fluctuations in the occupation numbers.* It is known that the instantaneous value $\hat{n}_p$ of the fermion occupation number is zero or one. (Instantaneous values of fluctuating quantities will be denoted by a caret, and average values by the brackets $\langle\langle\cdot\cdot\cdot\rangle\rangle$. Hence, $\langle\langle\hat{n}_p\rangle\rangle = n_p$.) Furthermore, different momentum states are statistically independent; thus, we arrive at the relation

$$\langle\langle\hat{n}_p\hat{n}_{p'}\rangle\rangle = \begin{cases} n_p & \text{if} \quad p = p' \\ n_p n_{p'} & \text{if} \quad p \neq p'' \end{cases} \tag{30}$$

This means that the control function is also a fluctuating quantity such that

$$\delta\hat{\chi} = -\int d\varepsilon_p \frac{1}{E_p}\,\delta\hat{n}_p \tag{31}$$

*The following considerations are based on Ref. 11.

where $\delta\hat{n}_p = \hat{n}_p - n_p$. Thus, we find the correlation function of the gap control

$$\langle\langle(\delta\hat{\chi})^2\rangle\rangle = \langle\langle\hat{\chi}^2\rangle\rangle - \langle\langle\hat{\chi}\rangle\rangle^2$$
$$= [2N(0)\Omega]^{-2} \sum_{p\sigma} \frac{1}{E_p^2} n_p(1-n_p) \tag{32}$$

where $[2N(0)\Omega]^{-1}\Sigma_{p\sigma} \cdots$ has been substituted for $\int d\varepsilon_p \cdots$, and where Ω is the volume of the system. Evidently, only states with E_p of the order of some Δ's contribute in Eq. (32), which allows us to put $n_p = 1/2$ with good accuracy. Hence

$$\langle\langle(\delta\hat{\chi})^2\rangle\rangle = \pi/[8N(0)\Omega\Delta] \tag{33}$$

Only collisions of the quasiparticles with phonons lead to fluctuations in the occupation number.* Hence, the inelastic collision time τ_E is the correlation time for $\delta\hat{n}_p$, as well as for $\delta\hat{\chi}$. Therefore, the power spectrum is given by

$$\langle\langle|\delta\hat{\chi}|^2\rangle\rangle_\omega = \frac{2\tau_E}{1+\omega^2\tau_E^2}\langle\langle(\delta\hat{\chi})^2\rangle\rangle \tag{34}$$

Since $\delta\hat{\chi}$ is a sum of a very large number ($\propto \Omega$) of independently fluctuating contributions, it constitutes a Gaussian process which is completely specified by Eq. (34). Insofar as $\Delta \ll T_c$, the order parameter varies slowly in time $\tau_{R\Delta} \sim \omega^{-1} \gg \tau_E$. This means that for the purpose we have in mind, we may consider $\delta\hat{\chi}$ as a white Gaussian noise.

As far as the space dependence of this noise is concerned, we should realize that fluctuations in the gap control at two different regions are, in essence, statistically independent if the two regions are farther apart than the thermal wavelength v_F/T_c (or the mean free path $v_F\tau_i$, if this is the smaller quantity) of the quasiparticles. Since the order parameter changes at a scale $\xi(T)$ which is much larger, we may assume that the fluctuations of $\delta\hat{\chi}$ are spatially uncorrelated. Thus we conclude, in accordance with Eqs. (33) and (34), that

$$\begin{gathered}\delta\hat{\chi}(r,t) = \left[\frac{\pi\tau_E}{4N(0)\hat{\Delta}}\right]^{1/2}\hat{\sigma}(r,t)\\ \langle\langle\hat{\sigma}(r,t)\hat{\sigma}(r',t')\rangle\rangle = \delta(t-t')\delta(r-r')\end{gathered} \tag{35}$$

This result allows us to generalize the time-dependent Ginzburg–Landau equation (19), augmented by the standard deformation energy, to

*Imagine that free particle states are used which are eigenfunctions of the impurity problem.

the following nonlinear Langevin equation:

$$\frac{\pi\tau_E\hat{\Delta}}{4T_c}\frac{\partial}{\partial t}\hat{\Delta}$$
$$= -\left[\alpha + \beta\frac{\hat{\Delta}^2}{T_c^2} - \xi^2(0)\nabla^2 - B\frac{\hat{\Delta}}{T_c}\right]\hat{\Delta} + \left[\frac{\pi\tau_E\hat{\Delta}}{4N(0)}\right]^{1/2}\hat{\sigma}(r, t) \qquad (36)$$

The mathematical form of this equation is different from the common type, since the time derivative is nonlinear and the Langevin force depends explicitly on the variable $\hat{\Delta}$. For convenience, we introduce a new variable $\propto \hat{\Delta}^{3/2}$, divide Eq. (36) by $\hat{\Delta}^{1/2}$, and thus obtain a standard Langevin equation.

It is not necessary here to write down the corresponding Fokker–Planck equation.* It is clear, however, that we may obtain the stationary probability distribution W_{st} by a simple integration. We now return to the original variable Δ, with the following result:

$$W_{st}(\Delta) = \text{const}\,(2\tau_E\Delta)^{1/2}\exp\left[-\mathscr{F}(\Delta)/T_c\right] \qquad (37)$$

where the prefactor follows from the substitution $d\Delta^{3/2} \propto \Delta^{1/2}d\Delta$. Furthermore,

$$\mathscr{F}\{\Delta\} = 2N(0)\int d^3r\,[\Phi(\Delta) + \tfrac{1}{2}\xi^2(0)(\nabla\Delta)^2] \qquad (38)$$

where

$$\Phi(\Delta) = \tfrac{1}{2}[\alpha + \tfrac{1}{2}\beta\Delta^2/T_c^2 - \tfrac{2}{3}B\Delta/T_c]\Delta^2 \qquad (39)$$

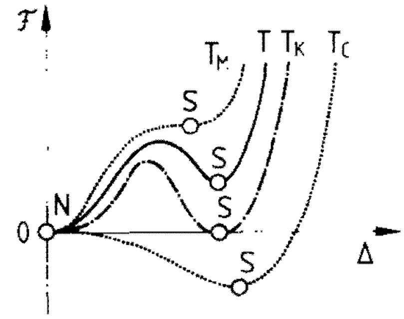
Obviously, the quantity $\mathscr{F}$ represents a generalization of the Ginzburg–Landau free energy to dissipative states† which agrees with the usual expression when $B = 0$. There is a complete equivalence of dissipative states with thermal equilibrium states. It follows that spatially homogeneous states have the lowest free energy, and hence, the largest probability. In the case of spatially homogeneous states, with $B \neq 0$ and $0 < \alpha < \alpha_M$, $\mathscr{F}$ has three stationary points corresponding to the three solutions of the equation $\partial\Phi/\partial\Delta = 0$. The two minima of $\mathscr{F}$ correspond to the two locally stable states, which are a normal and a superconducting state, whereas the maximum of $\mathscr{F}$ is associated with the unstable state. A qualitative graph of $\mathscr{F}$ as a function of Δ for different temperatures is shown in Fig. 3.

For any reasonable values of the volume Ω, the quantity $|\mathscr{F}/T_c|$ is a very large number. Consequently, the state that has the smallest value of $\mathscr{F}$ is

*For further information, see Ref. 12.

†In thermal equilibrium, the Langevin force can be found conveniently by evoking the dissipation–fluctuation theorem. The detailed construction of the Langevin force above makes it clear that this force is practically unaffected by the external perturbation.

Fig. 3. Free energy following from the probability distribution of Δ. Solid line: the normal state is globally stable. Dash-dotted line: coexistence of normal and superconducting states. Dotted lines: limit of local stability of the normal (respectively superconducting) state.



almost certainly realized. Obviously, this state is the globally stable state. Calculating the free energy explicitly, one finds that the superconducting (normal) state is stable if $\alpha < \alpha_K$ ($\alpha > \alpha_K$), where

$$\alpha_K = \tfrac{2}{9} B^2/\beta = \tfrac{8}{9}\alpha_M \tag{40}$$

At the temperature $T_K = T_c(1+\alpha_K)$, a discontinuous phase transition between the superconducting and the normal state takes place, which we call a first-order dissipative phase transition.

Let us now consider the deterministic Ginzburg–Landau equation, by which we mean Eq. (36) without the Langevin force. There exists a stationary solution ($\dot{\Delta} = 0$) at $T = T_K$, which shows that the normal and the superconducting phase, separated by a plane boundary layer, may coexist. The construction of such an essentially one-dimensional solution of a nonlinear differential equation has been discussed recently in connection with the soliton problem. For convenience, we give a survey of the relevant results in Appendix B. Furthermore, one can show that, for $T > T_K (T < T_K)$, the boundary layer moves with constant velocity [of the order $\xi(T)/\tau_E$] into the superconducting (normal) region.

In the same way as in thermodynamic first-order phase transitions, supercooling and superheating may occur here as well. The rather small probability for the formation of the energetically favourable phase is responsible for this. For illustration, let us calculate the lifetime of a metastable state for a one-dimensional sample* starting from the Langevin equation (36). Such a calculation was first performed by Kramers[13] on the basis of the equivalent Fokker–Planck equation for the case of a single degree of freedom and his approach was later generalized to systems with many degrees of freedom.[14,15] According to this theory, the lifetime for a decay of a metastable state Δ_m is mainly determined by transitions along those paths in the phase space—which is here the function space $\{\Delta(x)\}$—that traverse the lowest saddle point $\Delta_s(x)$ of the free energy functional $\mathcal{F}\{\Delta(x)\}$. This saddle point describes a superconducting (normal) droplet for temperatures $T_c < T < T_K$ ($T_K < T < T_M$).

*Sample with lateral dimensions smaller than $\xi(T)$.

If the free energy barrier

$$\mathscr{F}_b = \mathscr{F}\{\Delta_s(x)\} - \mathscr{F}\{\Delta_m\} \tag{41}$$

(the subscript m refers to the metastable state) is much larger than the thermal energy T_c, the transition occurs with an average rate

$$R = A \exp\left(-\mathscr{F}_b/T_c\right) \tag{42}$$

where the prefactor A is related to the dynamical behavior of the system in the vicinity of the saddle point and of the metastable state.

Let us first concentrate on $\Delta_s(x)$ and $\mathscr{F}_b$. The saddle point $\Delta_s(x)$ is a soliton-like solution of the stationarity condition (see Appendix B)

$$0 = -\partial\Phi(\Delta)/\partial\Delta + \xi(0)^2\Delta'' \tag{43}$$

where differentiation is denoted by a prime. It follows that $\Delta_s(x)$ obeys the relation

$$\tfrac{1}{2}\xi(0)^2(\Delta_s')^2 - \Phi(\Delta_s) = -\Phi(\Delta_m) \tag{44}$$

where $\Delta_s \to \Delta_m$ as $|x| \to \infty$. This allows us to calculate $\mathscr{F}_b$ without explicit knowledge of $\Delta_s(x)$ according to

$$\mathscr{F}_b = 4\sqrt{2}\, N(0)\xi(0)\mathscr{A}\left|\int_{\Delta_m}^{\Delta_n} d\Delta\, [\Phi(\Delta) - \Phi(\Delta_m)]^{1/2}\right| \tag{45}$$

where Δ_n is the order parameter next to Δ_m with $\Phi(\Delta_n) = \Phi(\Delta_m)$ and where $\mathscr{A}$ is the cross section of the sample.

The calculation of the prefactor A in Eq. (42) requires the knowledge of the eigenvalues of a Schrödinger equation, which is the deterministic Ginzburg–Landau equation (36) linearized around the saddle point $\Delta_s(x)$ and around the metastable state Δ_m, respectively. Whereas in the latter case the eigenvalues are easily calculated and turn out to be all positive, the eigenvalue problem connected with $\Delta_s(x)$ leads to a rather complicated Schrödinger equation. In order to arrive at an estimate for A despite our poor knowledge of the eigenvalue spectrum connected with $\Delta_s(x)$, we apply to our situation the conclusions which McCumber and Halperin have drawn from a detailed discussion of a corresponding calculation of the time scale of intrinsic resistive fluctuations in thin superconducting wires.[16] We thus propose that A is of the order of

$$A = \frac{L}{\xi(0)} \frac{1}{\tau_{R\Delta}} \tag{46}$$

In this relation, L is the length of the sample, and thus $L/\xi(0)$ measures the number of statistically independent subsystems. For $\tau_{R\Delta}$ we choose the gap relaxation time of a space-independent fluctuation near the metastable state. Thus $\tau_{R\Delta} \sim (T_c/\Delta)\tau_E$ for the superconducting, and $\tau_{R\Delta} \sim (T-T_c)^{-1}$ for the normal metastable state.

Calculating the free energy barrier, we find that as a function of α/α_M, it is proportional to B^3. It has a cusp at $\alpha = \alpha_K(T = T_K)$, where it assumes its maximal value

$$\mathscr{F}_b(\alpha_K) = 38.2N(0)\xi(0)\mathscr{A}T_c^2B^3 \tag{47}$$

Furthermore, $\mathscr{F}_b$ vanishes for $\alpha = 0$ and $\alpha = \alpha_M$, which indicates the limits of local stability at $T = T_c$ and $T = T_M$, respectively. (For illustration, see Fig. 6 for a graph of $\mathscr{F}_b$ in a similar problem.) Since $N(0)T_c\xi(0)\mathscr{A}$ is even in small samples, of the order of 10^6, we realize that metastable states will almost certainly not undergo any transition to the globally stable state except in a very small temperature range close to the limit of local stability.* Clearly, the value of the prefactor A is almost of no importance in this consideration.

5. QUASIPARTICLE DIFFUSION AND PHASE COEXISTENCE OF EMS

If $\Lambda \gg \xi(T)$, quasiparticle diffusion controls the space variation of the order parameter such that the Ginzburg–Landau deformation energy becomes unimportant. In spite of this complication, it is possible to find the conditions for the coexistence of metastable phases.

Since the distribution function $n_p = n_T + \eta_p$ as well as the order parameter Δ may depend on space, we start from the Boltzmann equation in the form (23), which contains the diffusion term $(|\varepsilon|/E)D\,\nabla^2\eta(E)$. The dependence of the prefactor $(|\varepsilon|/E)$ on the spatially varying order parameter requires, however, an approximation, and we propose to solve instead an equation of the following type:

$$(1-\Lambda^{*2}\nabla^2)\eta(E) = -\frac{B\Delta^2}{\pi T_c E} - n_T'\tau_E\frac{\Delta\dot{\Delta}}{E} \tag{48}$$

It is not difficult to show that this approximation leads to almost the same results in the space-dependent linear stability analysis, except that the function $h(\Lambda^2q^2)$ of Eq. (27) is replaced by $(1+\Lambda^{*2}q^2)^{-1}$. One may consult Fig. 2 in order to recognize that the choice

$$\Lambda^{*2} = 0.53\Lambda^2 \tag{49}$$

yields a rather good overall agreement.

*The same behavior can be observed in thermal equilibrium. See, for instance, Ref. 17.

For the moment, we neglect the time-dependent term and obtain for the control function

$$\chi = \int dE\, N_1(E) \frac{1}{E} (1 - \Lambda^{*2}\nabla^2)^{-1} \frac{B\Delta^2}{\pi T_c E} \tag{50}$$

Since the E integration involves only those Δ's on which the Laplacian ∇^2 does not operate, this integration can be performed without difficulty. Multiplying Eq. (50) from the left with Δ, we obtain

$$\chi \cdot \Delta = (1 - \Lambda^{*2}\nabla^2)^{-1} B\Delta^2/T_c \tag{51}$$

The time-dependent term of Eq. (48) leads to a contribution to the control function which can be calculated in the same way. Inserting these results in the Ginzburg–Landau equation, one finds it convenient to multiply it by $(1 - \Lambda^{*2}\nabla^2)$, upon which one obtains the following type of equation for the order parameter:

$$\frac{\pi\tau_E\Delta}{4T_c}\dot{\Delta} = -\left(\alpha + \beta\frac{\Delta^2}{T_c^2} - B\frac{\Delta}{T_c}\right)\Delta + \Lambda^{*2}\nabla^2\left(\alpha + \beta\frac{\Delta^2}{T_c^2}\right)\Delta \tag{52}$$

The peculiar form of a "deformation energy" has remarkable consequences. Let us first look for a solution of coexisting phases in the temperature range $0 < \alpha < \alpha_M$. We introduce formally a new order parameter

$$W = (\alpha + \beta\Delta^2/T_c^2)\Delta \tag{53}$$

and a new potential energy $\Psi(W)$, which is defined by $\partial\Psi/\partial W = \partial\Phi/\partial\Delta$. Obviously

$$\Psi(W(\Delta)) = \tfrac{1}{2}(\alpha + \beta\Delta^2/T_c^2)^2\Delta^2 - (\pi B\Delta^3/T_c)(\tfrac{1}{3}\alpha + \tfrac{3}{5}\beta\Delta^2/T_c^2) \tag{54}$$

Clearly, the extrema of Ψ are of the same type and at the same location as those of Φ. For stationary solutions $\dot{\Delta} = 0$, Eq. (52) can now be written in the form

$$-\partial\Psi/\partial W + \Lambda^{*2}\nabla^2 W = 0 \tag{55}$$

We conclude from this form that a solution of coexisting phases† exists if the potential Ψ has the same values in the normal and in the superconducting states. This is exactly the case if $T = \tilde{T}_K$, where $\tilde{T}_K$ is defined by

$$\tilde{\alpha}_K = \frac{15}{64}\frac{B^2}{\beta} = \frac{15}{16}\alpha_M \tag{56}$$

Note that $\tilde{\alpha}_K$ is different from α_K, though by only about 5%.

†Note that the spatial variation of this solution is on a scale of Λ^* independent of the magnitude of Δ. Hence, no expansion of χ in powers of $\Lambda^{*2}\nabla^2$ is possible.

For the sake of a clear argument, let us concentrate on the difference. One can show that for $T > \tilde{T}_K$ ($T < \tilde{T}_K$), the boundary layer moves with constant velocity into the superconducting (normal) region. Furthermore, for $T > \tilde{T}_K$, a stationary solution exists where a normal phase droplet of a well-defined size appears in a homogeneous superconducting phase. There are also time-dependent solutions, which show that any droplet of larger size increases indefinitely. Clearly, such behavior points to a first-order phase transition at $\tilde{T}_K$.

The significance of this result can be illuminated as follows. Assume for the moment that only spatially homogeneous states may exist. A Langevin equation of the type (36), adjusted to this case, would allow us to calculate transitions between the two reference states resulting in a relative probability which is characterized by a temperature of coexistence (equal probability) T_K which is different. Therefore, one must keep in mind what has quite recently been emphasized by Landauer,[18] namely, that one needs to consider *all* paths in phase space connecting the two reference states in order to calculate their relative probability. If one includes in this consideration only those paths that consist of spatially homogeneous states, one finds a relative probability characterized by T_K. On the other hand, the paths that traverse the mountain pass of local nucleation lead to a relative probability characterized by $\tilde{T}_K$. In a large system, however, the number of paths of the latter type is overwhelmingly larger than the number of paths of the former type. Hence, we expect $\tilde{T}_K$ to be closest to the actual transition temperature.

Thus, we have found a system where the relative probability of two competing states changes if additional interconnecting paths are included. Such a behavior can only be found in dissipative systems; it means a violation of the principle of detailed balance which is known to hold for thermal systems.[19] It is, of course, possible that detailed balance exists in a dissipative system; the case $\xi(T) \gg \Lambda$ may serve as an illustration.

We have not written down the Langevin equation that generalizes the deterministic equation (52). It is known that a Langevin equation of a system that violates detailed balance—no matter how simple it may be in its appearance—poses an almost intractable problem. In general, a stationary distribution function may be assumed to exist. One expects that this distribution function is strongly peaked. We have already expressed our opinion that the peak of the distribution function shifts rather discontinuously from one state to the other at a temperature which should be very close to $\tilde{T}_K$.

The preceding discussion gives additional support to the conjecture[18] that the transition temperature for large systems depends more on the details of the deterministic equation than on the precise nature of the noise that drives the fluctuations. Thus, we may speculate that additional noise coming from external sources may not change the transition temperature appreciably, though its influence on the lifetime of the metastable states may be significant.

6. DIFFUSIVE INSTABILITY OF NEMS

In order to create a situation where diffusive instability occurs, we change deliberately the sign of the drive term in the Elesin model.* Thus, in Eq. (10), the coefficient B is now negative; in order to make this change clear, we put $\bar{B} = -B$. As a result, a superconducting state solution exists only below T_c, where

$$\Delta/T_c = [(\bar{B}^2 - 4\beta\alpha)^{1/2} - \bar{B}]/2\beta \tag{57}$$

Clearly, there is a normal state solution $\Delta = 0$ for all temperatures.

The normal state is locally stable above T_c; but it is unstable below T_c against fluctuations of sufficiently small wave vector. As far as the superconducting state equation (57) is concerned, we examine the stiffness X as given by Eq. (29) and find that this state is certainly stable ($X > 0$) against fluctuations of zero wave vector. However, this state might be unstable against fluctuations of very large wave vector. Indeed,

$$X(q=\infty) = \alpha + 3\beta\Delta^2/T_c^2 \tag{58}$$

and hence, such an instability $[X(q=\infty) < 0]$ occurs if

$$\bar{B} > (2/\sqrt{3})(-\alpha\beta)^{1/2} \tag{59}$$

In this situation, we expect large variations of Δ, and hence we need to resort to a nonlinear equation. This can be obtained using the same arguments as in Section 5. However, we have to keep in mind that the Ginzburg–Landau deformation energy tends to stabilize the spatially homogeneous state, and therefore it has to be included in the present consideration. Thus, as a generalization of Eq. (52), we obtain

$$\begin{aligned}\frac{\pi\tau_E\Delta}{4T_c}\dot{\Delta} = &-\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - \xi^2(0)\nabla^2 + \bar{B}\frac{\Delta}{T_c}\right]\Delta \\ &+ \Lambda^{*2}\nabla^2\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - \xi^2(0)\nabla^2\right]\Delta\end{aligned} \tag{60}$$

In the following, we will discuss only the comparatively simple situation where $\Delta/T_c \ll (-\alpha/\beta)^{1/2}$ such that the cubic terms can be neglected. This

*For a possible realization of NEMS, see the discussion at the end of Appendix F.

allows us to write

$$(\pi\tau_E\Delta/4T_c)\dot{\Delta} = -\delta U/\delta\Delta \tag{61}$$

where the potential function is given by

$$U = \frac{1}{2}\int d^3r\left\{\left(\alpha + \frac{2\bar{B}}{3}\frac{\Delta}{T_c}\right)\Delta^2 + [\alpha\Lambda^{*2} + \xi^2(0)](\nabla\Delta)^2 + \Lambda^{*2}\xi^2(0)(\nabla^2\Delta)^2\right\} \tag{62}$$

There appears a contribution to the deformation energy $\propto(\nabla\Delta)^2$, which is negative ($\alpha<0$) for $\Lambda^*>\xi(T)$, indicating the diffusive instability. However, for larger values of the gradients, the deformation energy becomes positive on account of the term $\propto(\nabla^2\Delta)^2$.

According to Eq. (61), the motion of Δ is such that it approaches the state of minimal U (cf. the discussion at the end of Appendix B). The linear stability analysis reveals that the spatially homogeneous state is locally stable for $T>T_m$, where T_m is defined by $\alpha_m = -5.83\xi^2(0)/\Lambda^{*2}$, and that at T_m, an instability develops with respect to fluctuations of finite wave vector $q(\alpha_m)$, where

$$q^2(\alpha) = \frac{1}{2}\left(\frac{|\alpha|}{\xi^2(0)} - \frac{1}{\Lambda^{*2}}\right) \tag{63}$$

As far as the general state of minimal U is concerned, it seems hardly possible to find an exact solution of a nonlinear equation which is of fourth order in the space derivative. Therefore, we use a variational ansatz

$$\Delta_v = g(1+\mu\cos qx); \qquad |\mu|\leq 1 \tag{64}$$

which is exact in the limit $\mu\to 0$. Thus, we obtain a potential $U_v(g,\mu,q)$. It is easy to show that the homogeneous solution ($\mu=0$) is a local minimum for U_v in the temperature range $T_m<T<T_c$. In addition, for all temperatures below T_M given by $\alpha_M = -3\xi^2(0)/\Lambda^{*2}$, there exists a local minimum which corresponds to a space-dependent solution with $\mu=1$ and a wave vector q defined by Eq. (63). In the intermediate temperature range $T_m<T<T_M$, we also find a branch of saddlepoint solutions where μ changes gradually with increasing temperature from zero to one, thus connecting continuously the two branches of local minima. This situation is qualitatively similar to the cases discussed earlier. Comparing the value of the potential U_v in the two local minima, we find that a first-order phase transition takes place at a temperature $T_K(T_m<T_K<T_M)$ given by $\alpha_K = -4.51\xi^2(0)/\Lambda^{*2}$. For temperatures below T_K, which means for

$$\Lambda^* > \Lambda_K^* = 2.21\xi(T) \tag{65}$$

the space-dependent state ($\mu = 1$) is realized. This state may be described roughly as a succession of normal and superconducting layers where the excess quasiparticles created in the superconducting regions migrate in the adjacent normal layers. The intrinsic period of this structure is essentially equal to $\sqrt{2}\,\xi(T)$. Furthermore, the maximal order parameter $2g$ increases with decreasing temperature, and, at the transition point, it is larger by a factor 1.5 than its value $|\alpha|T_c/\bar{B}$ in the spatially homogeneous state.

7. MICROWAVE-STIMULATED SUPERCONDUCTIVITY

It has been pointed out by Eliashberg[20] that the pair blocking excitations of a superconductor can be removed from the gap edge by any radiation provided its frequency ν is small compared to T_c. Thus, it is possible to stimulate superconductivity in this way.

In general, radiation quanta generate as well as scatter the quasiparticles and thus give rise to a rather complicated drive term in the Boltzmann equation. We consider presently the case of electromagnetic radiation. This specifies the type of the BCS coherence factor and we obtain

$$(\dot{n}_p)_D = \frac{N_R}{2N(0)\nu}\left\{N_1(E-\nu)\left[1+\frac{\Delta^2}{E(E-\nu)}\right]\right.$$
$$\left.\times[n(E-\nu)-n(E)]+[\nu\to-\nu]\right\} \tag{66}$$

where N_R is the number of quanta absorbed* per unit volume and unit time. Note that quasiparticles are generated in the energy range $\Delta < E < \nu - \Delta$ only when $\nu > 2\Delta$, and none otherwise. Assuming the radiation to be weak, we start with the ansatz $n_p = n_T + \eta$, where η is small. Then, in leading order, we encounter in Eq. (66) expressions of the form $n_T(E-\nu) - n_T(E)$, which we approximate by $-n_T'\nu$ since $\nu \ll T_c$ by assumption.† Thus,

$$(\dot{n}_p)_D = \frac{1}{\tau_E} B\left\{N_1(E-\nu)\left[1+\frac{\Delta^2}{E(E-\nu)}\right]-(\nu\to-\nu)\right\} \tag{67}$$

where

$$B = \tau_E N_R / 8N(0)T_c \tag{68}$$

One easily recognizes that $(\dot{n}_p)_D$ vanishes if $\Delta = 0$, and that, as a function of energy, $(\dot{n}_p)_D$ differs from zero only in the narrow energy range $E \sim O(\Delta) \ll T_c$. This justifies the approximation $-n_T'\nu = \nu/4T_c$.

In the stationary, spatially homogeneous case, we have $\eta = \tau_E(\dot{n}_p)_D$. The result of the calculation of the order parameter control will be put in the

*Precisely, the number of quanta that would be absorbed at the same strength of the electric field if the metal were in its normal state.

†See Appendix F for a discussion of next highest order effects.

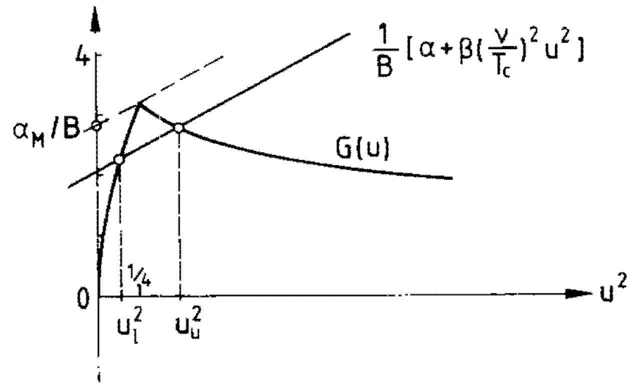


Fig. 4. Microwave stimulation. The intersection of the straight line $B^{-1}[\alpha+\beta(\nu/T_c)^2u^2]$ with G determines the solutions $\Delta_l=\nu u_l$ and $\Delta_u=\nu u_u$ on the lower and on the upper branches, respectively. Note the kink in G at $2\Delta=\nu$, which marks the onset of pair-breaking for $2\Delta<\nu$. For a given radiation power, there are superconducting solutions only if $\alpha<\alpha_M$, where α_M/B is marked on the ordinate.

form $\chi=BG(\Delta/\nu)$, where a graph of the function G is shown in Fig. 4. A detailed expression is given in Appendix C.

The normal state with $\Delta=0$ is always a solution of the Ginzburg–Landau equation. Superconducting state solutions with $\Delta\neq0$ can be found from a graphical construction as shown in Fig. 4. There is one solution for $T<T_c$, two solutions for $T_c<T<T_M$, and none for $T>T_M$, where T_M is defined by [here and in the following, we consider only the important case $B>0.1\beta(\nu/T_c)^2$]

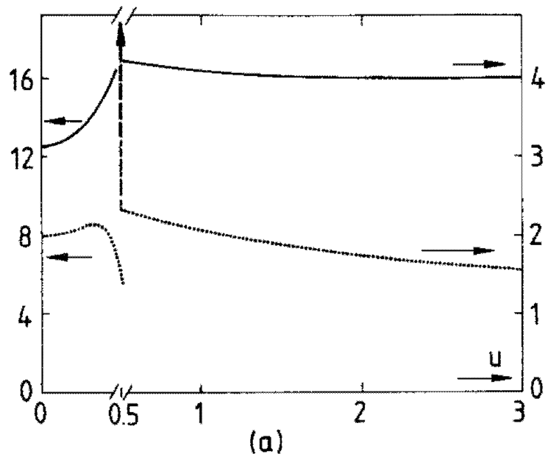
$$\alpha_M=[(2\pi/\sqrt{2})B-\tfrac{1}{4}\beta(\nu/T_c)^2] \tag{69}$$

The dependence of the order parameter on the temperature is qualitatively the same as shown in Fig. 1.

In order to investigate local stability, we consider first the case of spatially homogeneous fluctuations and proceed in the same way as in Section 3. Thus, we end up with an equation which agrees with Eq. (18) except that in the static part the term $B\Delta/T_c$ is replaced by $BG(\Delta/\nu)$. It can easily be inferred from Fig. 4 that $-(\partial/\partial\Delta)[\alpha+\beta(\Delta^2/T_c^2)-BG]\Delta$ is negative (positive) for the upper (lower) branch states, which are, consequently, stable (unstable) states.

In the next step, we investigate the stability against spatially inhomogeneous fluctuations and obtain a type of equation similar to Eq. (29). We parametrize X as follows:

$$X(\Delta;\,\Lambda^2q^2)=\Delta\left[\frac{\partial}{\partial\Delta}\left(\alpha+\beta\frac{\Delta^2}{T_c^2}\right)+\frac{1}{\nu}Bz\left(\frac{\Delta}{\nu};\,\Lambda^2q^2\right)\right] \tag{70}$$



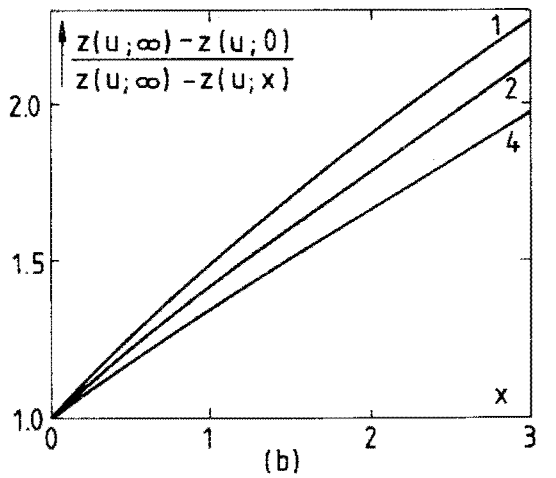


Fig. 5. (a) Microwave stimulation. Plot of $z(u;\infty)-z(u;0)$ (solid line) and of $\partial z(u;x)/\partial x$ at $x=0$ (dotted line) as a function of $u=\Delta/\nu$. Note that the scale is different for $0<u<1/2$ and for $u>1/2$. (b) Plot of $[z(u;\infty)-z(u;0)]/[z(u;\infty)-z(u;x)]$ as a function of x. The values of the parameter $u=\Delta/\nu$ are 1, 2, and 4.

Computing $z(u;x)$, we encounter an integral of the form shown in Eq. (25). Its evaluation is rather tedious due to the two singular density-of-state factors which appear in the integrand. Also, no simple approximation is possible since the integrand changes sign, which leads to a substantial cancellation. Some of the results we have obtained analytically are presented in Appendix C. In addition, we have also performed some integrations numerically. The general behavior of $z(u;x)$ is shown in Fig. 5. Accordingly, $z(u;\infty)-z(u;0)$ is positive and approximately 4 and 15 in the ranges $u>1/2$ and $u<1/2$, respectively. In addition, $\partial z(u;x)/\partial x$ at $x=0$ is roughly 2 and 8 in the ranges $u > 1/2$ and $u < 1/2$, respectively. Furthermore, $[z(u;\infty) - z(u;0)]/[z(u;\infty) - z(u;x)]$ is, in the range $x \leqslant 3$, approximately a linear function of x with slope 0.4.

Thus, in the case of a radiation-stimulated superconductor, we come to conclusions similar to those found for EMS: upper (lower) branch states are locally stable (unstable), and quasiparticle diffusion leads to an increase in stiffness against fluctuations of increasing wave number. This result is remarkable insofar as the drive term in the Boltzmann equation has a rather complicated energy dependence, in contrast to EMS.

In analogy to the discussion in Section 4, which concerns phase transitions, we consider first the case $\xi(T) \gg \Lambda$, where quasiparticle diffusion may be neglected. We include in the equation of motion for the order parameter the quantum fluctuations of the occupation numbers, the usual deformation energy $\xi^2(0)\,\nabla^2\Delta$, and obtain a time- and space-dependent Langevin equation. The further arguments one uses are identical with those given in the corresponding paragraphs of Section 4. The Langevin equation which follows is the same as Eq. (36) except for the obvious replacement of $B\hat{\Delta}/T_c$ by $BG(\hat{\Delta}/\nu)$. Consequently, the probability distribution of Δ defines a free energy of the form (38), where the potential

$$\Phi(\Delta) = \int_0^\Delta d\Delta' \left[\alpha + \beta(\Delta'^2/T_c^2) - BG(\Delta'/\nu)\right]\Delta' \tag{71}$$

now appears in place of Eq. (39). We conclude that spatially homogeneous states have the lowest free energy and that at a definite temperature T_K, $T_c < T_K < T_M$, a first-order phase transition between the superconducting and the normal states occurs. In the limit of $B \to \infty$, we have $\alpha_K/\alpha_M = 0.79$; in general, α_K/α_M decreases with the reduced radiation strength B, but not by more than 10%.

Obviously, at $T = T_K$ a stationary solution of the deterministic Ginzburg–Landau equation exists, which shows that the normal and the superconducting phases, separated by a plane boundary layer, may coexist. Also, we encounter here the phenomena of superheating and supercooling. The lifetime of the metastable state can be calculated as in Section 4, and it can be expressed by a decay rate of the form of Eq. (42). The height of the barrier $\mathscr{F}_b$ plays an essential role and we show a graph of $\mathscr{F}_b$ in Fig. 6 for two values of B. Again, we find that the lifetime is excessively large under ideal conditions.

We investigate now the consequences of quasiparticle diffusion, which is important in the case $\Lambda \gg \xi(T)$. The problem here is more involved than the one of EMS studied in Section 5. Therefore, we construct a model which allows us to understand at least some of the physics. To the extent that the

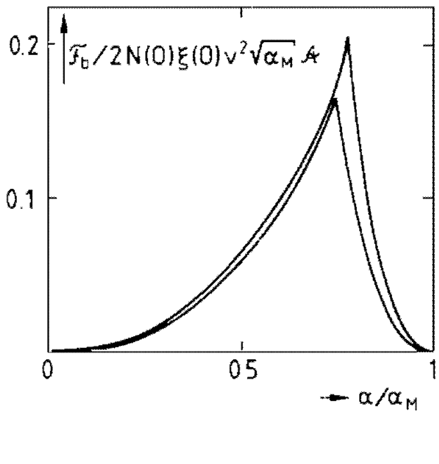


Fig. 6. Microwave stimulation. Plot of the reduced energy barrier $\mathcal{F}_b/2N(0)\nu^2\xi(0)\sqrt{\alpha_M}\mathcal{A}$ as a function of α/α_M calculated for the two values $B/\alpha_M = 0.28$ (upper curve) and $B/\alpha_M = 0.31$ (lower curve).

graphs of Fig. 5b may be considered as straight lines with almost common slope, we may approximate the result of the linear stability analysis on the quantity X as follows:

$$X = \Delta\left\{\frac{\partial}{\partial\Delta}\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - BG\left(\frac{\Delta}{\nu}\right)\right] + \frac{B}{\nu}\frac{x}{1+x}\left[z\left(\frac{\Delta}{\nu};\infty\right) - z\left(\frac{\Delta}{\nu};0\right)\right]\right\} \tag{72}$$

where $x = \Lambda^{*2}q^2$. As a model, we propose now the following nonlinear equation for the order parameter*:

$$\frac{\pi\tau_E}{4T_c}\frac{1}{1-\Lambda^{*2}\nabla^2}\Delta\dot{\Delta} = -\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - BG\left(\frac{\Delta}{\nu}\right)\right]\Delta - B\frac{-\Lambda^{*2}\nabla^2}{1-\Lambda^{*2}\nabla^2}H\left(\frac{\Delta}{\nu}\right)\Delta \tag{73}$$

where

$$H\left(\frac{\Delta}{\nu}\right)\Delta = \int_0^\Delta d\Delta'\,\frac{\Delta'}{\nu}\left[z\left(\frac{\Delta'}{\nu};\infty\right) - z\left(\frac{\Delta'}{\nu};0\right)\right] \tag{74}$$

If linearized around a stationary homogeneous state, Eq. (73) is equivalent to Eq. (29) with X given by Eq. (72). Of course, the above generalization is by no means unique. In the next step, we consider stationary solutions and put Eq. (73) in the following form:

$$-\partial\Psi/\partial W + \Lambda^{*2}\,\nabla^2 W = 0 \tag{75}$$

*Strictly, the Λ^*'s occurring on both sides are defined somewhat differently. In the following, we will draw definite conclusions only from the stationary equation, in which case we may put $\Lambda^{*2} = 0.4\Lambda^2$.

where

$$
\begin{aligned}
W &= [\alpha + \beta\Delta^2/T_c^2 - BG(\Delta/\nu) + BH(\Delta/\nu)]\Delta \\
\partial\Psi/\partial W &= [\alpha + \beta\Delta^2/T_c^2 - BG(\Delta/\nu)]\Delta
\end{aligned} \tag{76}
$$

Taking into account the properties of $G(\Delta/\nu)$ and $z(\Delta/\nu;\infty)$ [they are given by Eqs. (C2) and (C4) of Appendix C], one can show that $(H-G)\Delta$ vanishes if $\Delta < \nu/2$, and that one may put approximately

$$
[H(\Delta/\nu) - G(\Delta/\nu)]\Delta = 2\nu(\Delta/\nu - \tfrac{1}{2})^2; \qquad \Delta > \nu/2 \tag{77}
$$

In order to calculate the temperature $\tilde{T}_K$ for the coexistence of the normal and the superconducting states, we proceed as in Section 5. We obtain $\tilde{\alpha}_K/\alpha_M = 0.80$ for $B = \infty$; as B decreases, $\tilde{\alpha}_K/\alpha_M$ behaves similarly to α_K/α_M. For the same reasons as mentioned in the preceding section, we consider $\tilde{T}_K$ as the transition temperature of a large sample. In general, all qualitative conclusions remain the same.

8. PHONON-STIMULATED SUPERCONDUCTIVITY

Microwave and phonon stimulation differ formally in the BCS coherence factors. This has important consequences on the generation of the quasiparticles in the pair-breaking case $\nu > 2\Delta$. The generation rate is reduced in the microwave case, but it is considerably increased in the phonon case. Accordingly, Eq. (67) of the preceding section has to be replaced by

$$
(\dot{n}_p)_D = \frac{1}{\tau_E} B\left\{N_1(E-\nu)\left[1 - \frac{\Delta^2}{E(E-\nu)}\right] - (\nu \to -\nu)\right\} \tag{78}
$$

Again we introduce the gap control function in the form $\chi = BG(\Delta/\nu)$. A graph of this function is shown in Fig. 7 (cf. Appendix D); we wish to emphasize the large, negative value of G in the pair-breaking regime and its abrupt change at $\Delta = \nu/2$. Superconducting state solutions with $\Delta \neq 0$ can be found from the graphical construction displayed in this figure. There is one solution for $T < T_m$, three solutions for $T_m < T < T_c$, two solutions for $T_c < T < T_M$, and none for $T_M < T$. Thereby, the characteristic temperatures T_m and T_M are defined by the relations

$$
\alpha_M = 2\pi\left(1 - \frac{1}{\sqrt{3}}\right)B - \frac{\beta}{4}\left(\frac{\nu}{T_c}\right)^2; \qquad \alpha_m = -\left[\frac{2\pi}{\sqrt{3}}B + \frac{\beta}{4}\left(\frac{\nu}{T_c}\right)^2\right] \tag{79}
$$

and, furthermore, we have assumed B to be so large that $\alpha_M > 0$. The dependence of Δ on the temperature is sketched in Fig. 8.

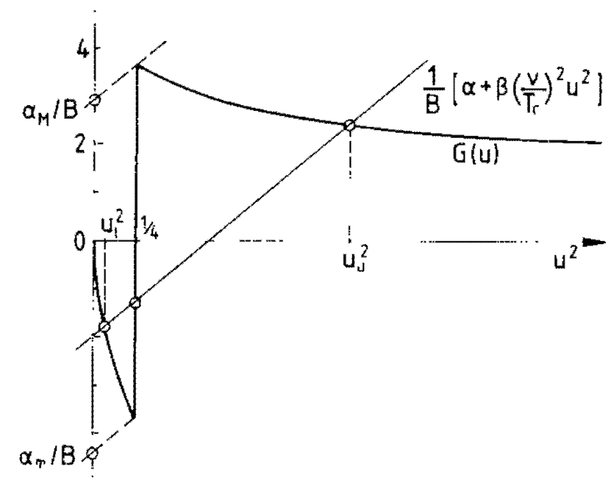


Fig. 7. Phonon stimulation. The intersection of the straight line $B^{-1}[\alpha+\beta(\nu/T_c)^2u^2]$ with G determines the solutions $\Delta_l=\nu u_l$ and $\Delta_u=\Delta u_u$ on the upper and lower branches of Fig. 8, respectively. In addition, there is a solution $\Delta=\nu/2$. Note the construction of the characteristic quantities α_M and α_m.

Only the branch $\Delta=\nu/2$ is unstable against all fluctuations, whereas the remaining two are at least stable in the spatially homogeneous case. If there are space-dependent fluctuations, it is convenient to define a quantity $z(\Delta/\nu;\ \Lambda^2q^2)$ as previously in Eq. (70). Some analytical results for $z(u;x)$ are listed in Appendix D. One finds that $z(u;0)=(\partial/\partial u)G(u)$ is positive for all values of u. In the range $u>1/2$, we have $z(u;1)$ about equal to $z(u;0)$, and $z(u;\infty)$ about half this value. Hence, the states with $\Delta>\nu/2$ are definitively locally stable, though quasiparticle diffusion reduces the stiffness with increasing wave vector.

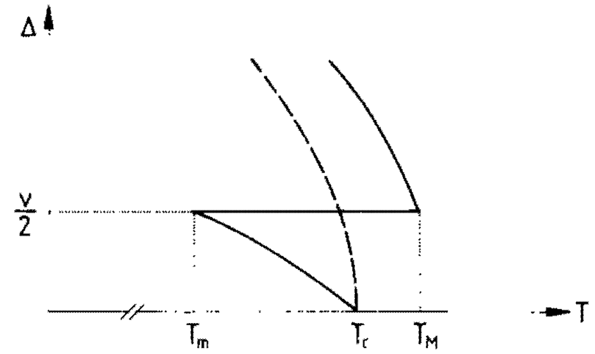


Fig. 8. Phonon stimulation. Temperature dependence of the energy gap (dashed line: thermal equilibrium).

Most interesting is the case $\Delta < \nu/2$, which is of importance if $\alpha < 0$. Both $z(u;0)$ and $z(u;1)$ are positive; but in most cases $z(u;1)$ is considerably smaller than $z(u;0)$. However, $z(u;\infty)$ is always negative, which indicates a strong tendency toward a diffusive instability. This tendency obviously has its origin in the strong pair-breaking properties of the phonons. Using the analytical expressions for $G(u)$ and $z(u;\infty)$, one can show that there is a diffusive instability for any value of B provided that $|\alpha|$ is sufficiently small. Furthermore, if $B > (\beta/2\pi)(\nu/T_c)^2$, such an instability exists for any value of α (which otherwise has to be in the range $0 > \alpha > \alpha_m$).

This property leads to spatially modulated states similar to the ones found for NEMS. In order to show this, we construct a nonlinear equation for Δ similar to Eq. (73) of the preceding section. However, we have to restrict the range of Δ to less than, say, 0.4ν in order that the quantity $[z(u;\infty)-z(u;0)]/[z(u;\infty)-z(u;x)]$ be, in a reasonable approximation, a linear function of x with a slope independent of u. (The slope is then again 0.4 as in the related situation displayed in Fig. 5b).) For the same reason as in NEMS, it is also necessary to include the stabilizing Ginzburg–Landau deformation energy in the consideration. Thus, we obtain

$$\frac{\pi\tau_E\Delta}{4T_c}\dot{\Delta} = -\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - \xi^2(0)\nabla^2 - BG\left(\frac{\Delta}{\nu}\right)\right]\Delta$$
$$+\Lambda^{*2}\nabla^2\left[\alpha + \beta\frac{\Delta^2}{T_c^2} - \xi^2(0)\nabla^2\right]\Delta \qquad (80)$$

In analogy to the discussion of the related equation (60), we restrict ourselves to the case where Δ is small. Then, $BG(\Delta/\nu) = -2\pi B\Delta/\nu$, and the cubic terms in Eq. (80) can also be neglected. There is now a complete correspondence between the present problem and NEMS; and Eqs. (61) and (62) apply if we make the identification $\bar{B} = 2\pi BT_c/\nu$. We conclude that a spatially modulated state is stationary and stable if $\Lambda^* > \Lambda_K^*$ [cf. Eq. (65)]. This is equivalent to the condition $\Lambda > \Lambda_K$, where $\Lambda_K = 3.4\xi(T)$.

Since the above analysis requires sufficiently small values of Δ, we are, however, not in a position to draw any conclusions about the global stability of these states with respect to the competing states $\Delta > \nu/2$.

Next, we consider the case $\xi(T) \gg \Lambda$, where quasiparticle diffusion is unimportant and where the methods of Section 4 can be applied. The Langevin equation for the order parameter defines a stationary probability distribution with a free energy of the form (38), where the potential Φ is the same as given by Eq. (71), however, with the appropriate G function now inserted. We conclude that spatially homogeneous states have the lowest free energy, and that at a given definite temperature T_K, $T_m < T_K < T_M$, a

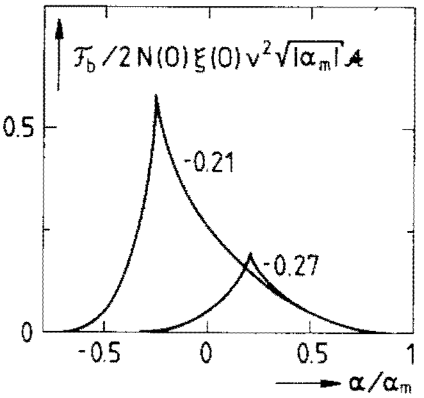


Fig. 9. Phonon stimulation. Plot of the reduced energy barrier $\mathcal{F}_b/2N(0)\nu^2\xi(0)|\alpha_m|^{1/2}\mathcal{A}$ as a function of α/α_m calculated for the two values $B/\alpha_m = -0.21$ and $B/\alpha_m = -0.27$.

first-order phase transition appears. The ratio α_K/α_m is only a function of

$$B/\alpha_m = -B/[(2\pi/\sqrt{3})B + \tfrac{1}{4}\beta(\nu/T_c)^2]$$

For increasing B, α_K/α_m decreases monotonically from $\alpha_K/\alpha_m = 1$ for $B = 0$ to $\alpha_K/\alpha_m = -0.322$ for $B = \infty$. We have $\alpha_K = 0$ if $B/\alpha_m = -0.244$. The phase transition occurs between a superconducting and a normal state for $T_K > T_c$, and between two superconducting states for $T_K < T_c$.

Obviously, at $T = T_K$, the two phases, separated by a plane boundary layer, may coexist. Also, superheating and supercooling may occur easily. A calculation of the free energy barrier $\mathcal{F}_b$ of the metastable states yields about the same values as in the case of microwave stimulation. We show a graph of $\mathcal{F}_b$ in Fig. 9 for two values of B, which correspond to the cases $T_K > T_c$ and $T_K < T_c$.

9. STIMULATION BY TUNNELING AND THE COEXISTENCE OF TWO ENERGY GAPS

The quasiparticle tunneling processes in a superconducting tunnel junction (SIS') also strongly modify the quasiparticle distribution. For example, by coupling two superconductors with different energy gaps $\Delta < \Delta'$, for values of the applied voltage $|eV| \approx \Delta' - \Delta$ excitations can be extracted out of the superconductor with the smaller gap, and hence, the superconductivity in this part of the junction can be stimulated. This effect has been described and detected experimentally by Chi and Clarke.[21] On the other hand, if the voltage $|eV|$ is larger than the sum of the gaps $\Delta + \Delta'$, the current of excitations changes drastically. This leads to a positive feedback since the increased number of excitations in the superconductor for $|eV| > \Delta + \Delta'$ lowers the value of the gap. By controlling the total injection current a coexistence of different gaps can be achieved. This effect was first detected by Dynes *et al.*[22] and this section is devoted to its discussion.*

*The basis of this discussion is the work of Ref. 23.

We consider a tunnel junction consisting of the superconductor of interest (the probe) coupled to a second superconductor (the generator). The gap of the generator Δ_G, in contrast to the gap of the probe, is supposed to be strong enough that it is not appreciably perturbed by the tunneling processes. The current of excitations (which differs from the electric current, since electron- and hole-like excitations contribute with different relative sign to the two currents) generates a drive term in the Boltzmann equation for the probe which has the form

$$(\dot{n}_p)_D = \frac{1}{4e^2 R\Omega N(0)}\{N_1^G(E-eV)[n_T(E-eV)-n_T(E)] \\ +(eV \leftrightarrow -eV)\} \tag{81}$$

The resistance of the junction is R; Ω and $N(0)$ are the (effective) volume and the density of states of the probe, and N_1^G is the BCS reduced density of states of the generator. For reasons already explained in the discussion following Eq. (66), corrections to the thermal distribution n_T have been neglected. Furthermore, we assume that Δ as well as Δ_G are much smaller than T_c. In this limit it is sufficient to consider small voltages $|eV| \ll T_c$.

The calculation of the gap control proceeds in a manner analogous to the cases discussed earlier in this paper. To lowest order in an expansion in $|eV|/T_c$ we obtain

$$\chi = BG(\Delta/\Delta_G, |eV|/\Delta_G) \tag{82}$$

where $B = \Delta_G \tau_E/16 T_c e^2 R\Omega N(0)$. A detailed expression for $G(u, v)$ is given in Appendix E. For the present problem, we are mainly interested in voltages $|eV| \approx \Delta_G + \Delta$. It turns out that the combined effect of extraction and injection of electron- and hole-like excitations on the control vanishes for voltages $|eV|$ larger than $\Delta_G + \Delta$. At $|eV| = \Delta_G + \Delta$ the control function is discontinuous and a step structure results. In the vicinity of this step we can approximate G by

$$G(u, v) = \pi v/\sqrt{u}\, \theta(u+1-v) \tag{83}$$

Again we employ a graphical construction to find the solutions of the Ginzburg–Landau equation; see Fig. 10. Depending on the value of the voltage, we find an enhanced gap (for $|eV| < \Delta_G + \Delta$) or an unperturbed gap (for $|eV| > \Delta_G + \Delta$, where χ is zero). For suitable intermediate values of the voltage we find three superconducting solutions Δ_1, Δ_2, and Δ_3, where $\Delta_G + \Delta_1 < \Delta_G + \Delta_2 = |eV| < \Delta_G + \Delta_3$. In addition, the normal state is always a solution. These values for the gap and the tunnel current density as a function of the applied voltage are shown in Fig. 11. Obviously, the two branches with Δ larger and smaller than $|eV| - \Delta_G$ extend to smaller and

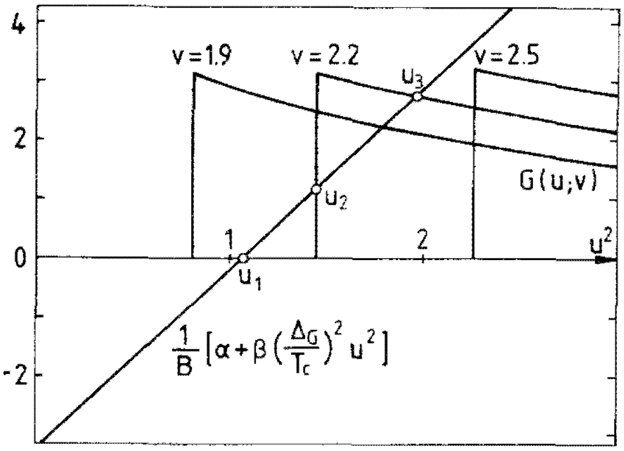


Fig. 10. Graphical solution of the Ginzburg–Landau equation for perturbation by tunneling of quasiparticles. The different forms of G correspond to different values of the applied voltage.

larger values of $|eV|$, respectively. In the region where the branches overlap, significantly differing values of the current density are also obtained.

The linear stability analysis is comparatively simple, since the drive term (81) depends only on the gap Δ_G, which is constant. Thus, of the two terms on the right side of the Boltzmann equation (23), only the second one acquires a fluctuating increment. Including the Ginzburg–Landau deformation energy, we obtain

$$-\frac{\pi\Delta}{4T_c} i\omega\tau_E h(\Lambda^2 q^2)\,\delta\Delta = -\left\{\frac{\partial}{\partial\Delta}\left[\alpha+\beta\frac{\Delta^2}{T_c^2}-BG\right]\Delta+\xi^2(0)q^2\right\}\delta\Delta \tag{84}$$

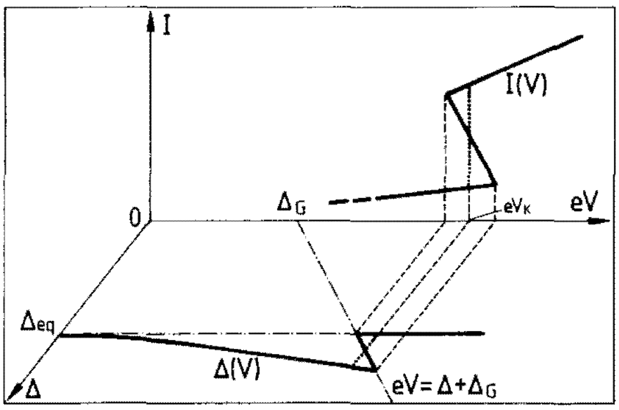


Fig. 11. The solutions Δ shown as a function of the applied voltage as obtained by the construction in Fig. 10. The injection current density corresponding to these solutions is qualitatively plotted.

It is obvious that the states with $\Delta = 0$ and Δ_2 are unstable. On the other hand, the states with Δ_1 and Δ_3 are stable for all wave vectors. Furthermore, we find that fluctuations with finite wave vector decay faster than homogeneous fluctuations.

Due to the properties of the drive term, as discussed above, it is not difficult to construct a nonlinear equation for Δ. In particular, stationary solutions obey the equation

$$0 = -\left(\alpha + \beta \frac{\Delta^2}{T_c^2} - BG\right) \Delta + \xi^2(0)\, \nabla^2 \Delta \tag{85}$$

which is exact for arbitrary relations between Λ and $\xi(T)$. The first term of this equation can be written in the form $\partial\Phi/\partial\Delta$. The minima of the potential $\Phi(\Delta)$ correspond to the locally stable, homogeneous solutions discussed above. In the region where we found two locally stable solutions, $\Phi(\Delta)$ has two minima at Δ_1 and Δ_3. It is roughly of the form indicated in Fig. 12 (with η replaced by Δ). The values of $\Phi(\Delta)$ at the locations of the minima depend on the voltage. In particular, there exists one value of the voltage V_K where the two minima are on the same level. The standard construction of soliton-like solutions (see Appendix B) reveals that at this voltage a stable solution of Eq. (85) exists, which describes a wall separating two regions with the different gaps Δ_1 and Δ_3. We identify this voltage as the transition point, such that at voltages $|V| < V_K$ the system is completely in the state where the gap is equal to Δ_3 and such that at voltages $|V| > V_K$ it is in the Δ_1 state. We remark that the present problem can be attacked also with the methods developed in Section 4. One finds a probability distribution which shifts discontinuously at $|V| = V_K$, confirming the conclusions concerning a first-order phase transition.

The two solutions Δ_3 and Δ_1 satisfy $\Delta_G + \Delta_3 > |eV_K| > \Delta_G + \Delta_1$ and hence the injection current density is significantly different for these two states. By controlling the total tunneling current, the system can be forced to split into two regions with different gaps, the relative size of these regions being determined by the total current. This situation is analogous to a liquid–gas transition of a van der Waals gas, where the Maxwell construction determines the pressure of the transition from one phase to the other, and where the control of the total volume enforces the coexistence of both phases. However, although the total current (volume) is externally controlled, it is the voltage (pressure) which is equal in both phases; hence it is this variable which has to be introduced as the basic variable in the theory. While the tunnel junction is biased on a point of the vertical part of the current–voltage characteristic, which we identified as a Maxwell construction, the probe splits into regions with two different gap values.

10. DISCUSSION

In this paper, the stability of dissipative states has been investigated in three steps of increasing complexity: (i) The Boltzmann and the gap equation have been solved for stationary spatially homogeneous states; (ii) the local stability of these states has been examined in a linear stability analysis which among other things, shows whether quasiparticle diffusion increases or decreases the stability; (iii) the nonlinear equation for the order parameter has been solved in reasonable approximation, with various results. If there are two competing stable states, only one is stable against large fluctuations. As a function of temperature, the stability changes, which leads to the appearance of a first-order phase transition. On the other hand, if there are states which are locally unstable against fluctuations of finite wave vector, the instability may lead to a stable, but spatially structured, state.

Two important limits of the linear stability analysis can be put in a more general form. Assume that $n_p^{(st)} = n^{(st)}(E_p; \Delta)$ is the stationary solution of the Boltzmann equation for any value of Δ independent of space and time. Then, it has been found that a solution of the gap equation is locally stable against spatially homogeneous fluctuations if $X_0 > 0$, where*

$$X_0 = -\frac{\partial}{\partial \Delta}\left[\int d\varepsilon_p \frac{1}{2E_p}(1-2n_p^{(st)}) - \frac{1}{\lambda}\right]\Delta \tag{86}$$

We conjecture that this condition is valid for a rather large class of dissipative states. If there are various solutions of the gap equation, then the sign of X_0 alternates with, say, increasing Δ. Consider now the case of fluctuations in Δ which oscillate rapidly in space. Then, the quasiparticle diffusion prevents any change in the distribution function at a given energy. Stability now requires $X_\infty > 0$, where

$$X_\infty = -\left\{\frac{\partial}{\partial \Delta}\left[\int dE\, N_1(E)\frac{1}{2E}(1-2n^{(st)}) - \frac{1}{\lambda}\right]\Delta\right\}_{n^{(st)}=\text{const}} \tag{87}$$

It is important to note that the derivative has to be taken at constant distribution function.

Clearly, the distribution function $n^{(st)}$ is an important input in this theory.† In order to obtain an analytical form,we have assumed $\Delta \ll k_B T_c$ and the phonons to be in thermal equilibrium close to T_c. Among other things, this allows us to approximate the collision operator by the relaxation ansatz.

*The subscripts 0 and ∞ refer to the wave vector of the fluctuation. For illustration, see Eq. (29), where X is given in the case of EMS. In equilibrium, X_0 is essentially the second derivative of the BCS free energy.

†Chang and Scalapino[24] have calculated $n^{(st)}$ numerically for various situations, but only for fixed values of Δ.

Very frequently discussed is the μ^* model of Owen and Scalapino,[25] where $n_p = [\exp(E_p - \mu^*)/k_B T + 1]^{-1}$. Chang and Scalapino[26] have shown that the states of this model are locally unstable for some values of the parameter. It seems that Eqs. (86) and (87) lead to the same conclusion. An extension of this theory by Scalapino and Huberman[27] starts from the relationship $\mu^* \propto -(N_Q - N_C)^2 + \text{const}$, which is found near the instability point N_C of the quasiparticle number N_Q. The authors assume that there is a diffusive quasiparticle current $\propto \nabla\mu^*$, and that the divergence of this current can be used to correct the rate equations of Rothwarf and Taylor[28] for quasiparticle diffusion. A linear stability analysis yields the result that there are stationary states which are unstable against spatially modulated fluctuations of a nonzero wave vector. The nonlinear aspect of this model was later investigated by Hida,[29] and his results confirm the previous conclusions. From a phenomenological point of view, however, it is not clear whether one may assume a current $\propto \nabla\mu^*$ for states that are locally unstable, i.e., for states where $\partial\mu^*/\partial N_Q < 0$.

In his paper on diffusive instability, Smith[30] considers various models for the form of the distribution functions and points out the possibility of the occurrence of spatially modulated instabilities in some cases. He emphasizes the important fact that quasiparticles diffuse at constant energy.

A microscopic calculation on a possible spatial structure in the case of microwave stimulation has been performed by Ivlev.[31] The linear stability analysis there is based on essentially the same ideas as our approach. The result of a spatial instability arises erroneously from allowing the quasiparticles to diffuse at constant ε_p instead of at constant E_p.

Important results with regard to the stability problem are contained in a recent paper by Elesin.[32] There, the case of an optically irradiated superconductor has been studied with the phonon temperature fixed at zero temperature. The phenomena which occur there are very similar to those of EMS here. In particular, one obtains values of the gap as a function of the absorbed radiation quanta N_R similar to Fig. 1 with T replaced by N_R. As far as the general methods and results are concerned, there is agreement between Elesin's work and the corresponding sections of our paper,† except for the concept of a free energy in dissipative systems, which is missing there.

The coexistence of two phases in the presence of strong quasiparticle injection has been investigated by Hida.[33] The model is based on the Rothwarf–Taylor equation augmented by a diffusive term, and an attempt has been made to construct a soliton-like solution. This concept is very much in the spirit of our approach.

In a paper on the same subject, Welte[34] examines the local stability of spatially homogeneous states. This approach differs in that the injection

†There are differences in some details—for instance, in the treatment of the quasiparticle diffusion. These do not lead to changes in the qualitative conclusions.

current density is assumed to be fixed. On the basis of the Rothwarf–Taylor equations, it was found that there exist two states at a given current and that the state with the larger (smaller) value of the gap is locally stable (unstable; but at very small values of Δ, it is stable again).

In the investigations of this paper we have found two different types of behavior of superconductors driven away from thermal equilibrium: the stable states are homogeneous in space (first type), or the stable states are spatially modulated (second type). In a simple model calculation, at least, a decrease or an increase of the quasiparticle generation with increasing energy gap leads to the first or to the second type of behavior, respectively.

We find superconductors of the first type to be those which are either weakly illuminated, stimulated by microwaves, or excited by quasiparticle injection. For the sake of definiteness, let us consider the first two examples (similar phenomena occur in the third case). There exists a range of temperatures, say $T_c < T < T_M$, where there are three solutions of the gap equation. The state with the intermediate value of the gap is locally unstable, whereas the remaining two states are locally stable. (Quasiparticle diffusion works such that the stability increases with increasing wave vector of the fluctuations.) This situation has a counterpart in the mean field theory of thermodynamic phase transitions, e.g., in the van der Waals theory of real gases. The difference, however, is that we are dealing here with a dissipative, and not with a thermodynamic, system. In spite of this difference, it is possible to construct a theory with the same structure as thermodynamics provided that quasiparticle diffusion is negligible, i.e., $\xi(T) \gg \Lambda$. Essentially, the theory is based on a Langevin equation for the order parameter where the noise results from the quantum fluctuations of the occupation numbers. We obtain a critical temperature T_K, $T_c < T_K < T_M$, as a condition for a first-order phase transition between the superconducting and the normal state. Superheating and supercooling may exist, and we calculate the lifetimes of the metastable states. Complications arise if quasiparticle diffusion dominates, i.e., $\Lambda \gg \xi(T)$. In order to obtain a nonlinear equation for the order parameter, it is necessary to simplify the diffusion process considerably. Even so, it is not possible to solve the Langevin equation for the order parameter, since now the dissipative systems lack detailed balance, which is otherwise an inherent feature of thermodynamic systems. However, we are able to give reasons why a first-order phase transition still may exist, and we calculate a new transition temperature $\tilde{T}_K$, which is not very different from T_K. In general, it is difficult to stabilize the coexistence of the two phases in an experiment, as can be done in a liquid–vapor transition, for instance, by keeping the volume fixed. An example contrary to this rule is a superconductor with quasiparticles injected via a tunnel junction, where the total current controls the coexistence of phases.

The second type of behavior is represented—except for a model system—only by phonon-irradiated superconductors in the pair-breaking regime $\Delta < \hbar\nu/2$. (In general, phonon stimulation rather leads to behavior of the first type.) There a stable state appears where the order parameter oscillates in space, with an intrinsic period of about $\xi(T)$, between zero and a finite value. We may visualize this state as a succession of normal and superconducting layers, where the excess quasiparticles created in the superconducting regions migrate into the adjacent normal layers.

In the experiments on radiation-stimulated superconductivity[35–39] there is good evidence for the occurrence of a sharp transition between a normal and a superconducting state, though the presence of heating does not allow a quantitative comparison between experiment and theory. Superconductors with strong quasiparticle injection may be interpreted to show a first-order transition either into a state of two coexisting phases[22,40] (with different values of the gap) or into the normal state.[41] In contrast to the general type of behavior mentioned above, superconductors irradiated by lasers at low temperatures show a rather incomplete transition to the normal state[42,43]— a fact which one has tried to interpret as a transition which involves a spatially or a temporally varying intermediate state.* Quite recently, a variety of phenomena have been reported[44] which have been observed in tunnel junctions under different conditions.

Finally, we recall that in all our investigations, we have only considered changes in the magnitude of the order parameter and not in its phase. Since the stationary states here are without supercurrent, this is of no consequence in the linear stability analysis, and, presumably, also not in the following analysis of the nonlinear problem. A different situation is met if superconductors under the condition of a fixed total current are considered. The investigations of Skocpol *et al.*[45] and of Kramer and co-workers[46] on phase-slip centers in one-dimensional superconductors have shown that in such a case there are stable states which are periodic in time as well as in space.

APPENDIX A

In order to show that a drive term proposed in the Elesin model may occur, we review the case studied by Elesin,[7] where a superconductor is exposed to optical radiation. There the radiation quanta create quasiparticles uniformly in an energy range up to the quantum energy. The residual Coulomb interaction allows the formation of new quasiparticles; eventually

*It is not clear whether the intrinsic mechanism which leads to the behavior of the second type is present in these experiments. According to the theory of Ref. 32, such a behavior should not be expected.

they lose their remaining energy by phonon emission and recombine. Thus, a stationary state results.

We are interested only in the final stage of this process, where the phonon collisions are important. Impurity scattering relaxes most effectively any anisotropic part— if present—of the distribution function. Hence, the inelastic phonon collisions need to be considered only in connection with the isotropic part $\langle n_p \rangle$ of the distribution function, which is an even function of ε_p, and one obtains

$$I_{ep}\{\langle n_p\rangle\}$$

$$\begin{aligned} &= \pi \int_{\varepsilon}^{\infty} d\varepsilon' \left(1-\frac{\Delta^2}{EE'}\right)\mu(E-E')[n(1-n')N_{E'-E}-(1-n)n'(N_{E'-E}+1)] \\ &+ \pi \int_{0}^{\varepsilon} d\varepsilon' \left(1-\frac{\Delta^2}{EE'}\right)\mu(E-E')[n(1-n')(N_{E-E'}+1)-(1-n)n'N_{E-E'}] \\ &+ \pi \int_{0}^{\infty} d\varepsilon' \left(1+\frac{\Delta^2}{EE'}\right)\mu(E+E')[nn'(N_{E+E'}+1)-(1-n)(1-n')N_{E+E'}] \end{aligned} \tag{A1}$$

In the expression above, $\varepsilon = \varepsilon_p$; $E = E_p$; $n = \langle n_p \rangle$; etc.; and N_E is the thermal phonon distribution function. Furthermore, $\mu(E)$ is the effective phonon density of states (usually denoted by $\alpha^2 F/2$), which obeys the relation $\lambda = 4\int_0^{\infty} dE\,\mu(E)/E$ and which in the Debye model is proportional to $E|E|/\theta^2$. It follows from the expression (A1) that in equilibrium, the inelastic scattering time τ_E of the quasiparticles is given by

$$\frac{1}{\tau_E} = 2\pi \int_0^{\infty} d\varepsilon' \sum_{\pm} \left(1 \mp \frac{\Delta^2}{EE'}\right) \frac{\mu(E \mp E')}{sh[(E\mp E')/2T]}\,\frac{ch(E/2T)}{ch(E'/2T)} \tag{A2}$$

In a normal metal, $1/\tau_E$ is of the order of T^3/θ^2 or E^3/θ^2, whichever quantity is larger. In most cases, we take it to be a constant $\sim T_c^3/\theta^2$.

Consider first the metal when it is irradiated in its normal state. The quasiparticle generation appears in the Boltzmann equation as a source term, spread out over a wide range of energies, but of very small magnitude. Consequently, the leading part of the distribution function is the thermal distribution $n_T(E_p)$ and the small correction $n_0(E_p)$ can be found by solving the linearized Boltzmann equation. As a result

$$n_0(E) \sim \begin{cases} BE/T & \text{for} \quad E \ll T \\ BT^4/E^4 & \text{for} \quad E \gg T \end{cases} \tag{A3}$$

where B is a dimensionless quantity

$$B \sim \tau_E N_R / N(0) T_c \tag{A4}$$

proportional to the number N_R of the radiation quanta absorbed per unit volume and unit time. We will assume that $B \ll 1$. The steep decrease of n_0 at higher energies reflects the increasing rate of spontaneous phonon emissions with increasing energy. At lower energy, stimulated emission and absorption of thermal phonons is the dominating process.

By inspecting the Boltzmann equation carefully, one may convince oneself that the distribution function $n_T + n_0$ is a good solution also in the superconducting state with accuracy of the order Δ^2/T_c^2, provided that one neglects the coherence factors $(1 \mp \Delta^2/EE')$ in the collision integral. We correct the distribution function by putting

$$n_p = n_T + n_0 + n_1 \tag{A5}$$

where the correction n_1 is obtained as the solution of the Boltzmann equation in which a combination of the remainders $\mp\Delta^2/EE'$ of the coherence factors and of n_0 appears as an inhomogeneous term (drive term). Examining this drive term, one recognizes that only the contribution

$$(\dot{n}_p)_D = -\frac{\pi\Delta^2}{E_p}\int_0^\infty d\varepsilon' \frac{n_0(E')}{E'}\,\mu(E')\,\mathrm{cth}\,\frac{E'}{2T} \tag{A6}$$

leads to an effective control on the order parameter, and hence it is the only part that we need to retain. Taking Eqs. (A2)–(A4) into account, we obtain Eq. (10), where for convenience, the numerical coefficient in the definition (A4) of B has been adjusted.

In conclusion, we remark that the contribution of n_0 to the gap control is negative and proportional to B. Furthermore, it is independent of Δ with accuracy of the order Δ^2/T_c^2. This means that it is irrelevant for the question of the stability that is of concern in this paper. Hence, we neglect it. However, it must be kept in mind that its contributions appear as an effective change in the electronic temperature. In general, there are other effects (e.g., heating) which work in the same direction and which lead to some ambiguity in the comparison between experiment and theory.

APPENDIX B

In this appendix we review some properties of a field theory in one space and one time dimension. Let us assume that the dynamics of the real scalar field $\eta(x, t)$ is governed by the potential functional

$$U\{\eta\} = \int dx\, [\Phi(\eta) + \tfrac{1}{2}\xi^2(\eta')^2] \tag{B1}$$

in such a way that*

$$\dot{\eta} = -\delta U\{\eta\}/\delta\eta \tag{B2}$$

*The following conclusions remain unchanged if $\dot{\eta}$ is replaced by $f(\eta)\dot{\eta}$, where $f > 0$. In the investigations of the text, $f(\Delta) \propto (1 + 2\tau_E\Delta)$.

The stationary states $\eta_{st}(x)$ of the functional $U\{\eta\}$ are the time-independent solutions of Eq. (B2)

$$\xi^2\eta''_{st} = \partial\Phi(\eta_{st})/\partial\eta_{st} \tag{B3}$$

This is mathematically identical to a familiar mechanical problem, the motion of a particle with mass ξ^2 in a potential *minus* $\Phi(q)$, if one identifies $\eta_{st}(x)\rightarrow q(t)$. The qualitative structure of the stationary states $\eta_{st}(x)$ may therefore be discussed by making use of the relation

$$\tfrac{1}{2}\xi^2(\eta'_{st})^2 - \Phi(\eta_{st}) = -C \tag{B4}$$

the mechanical analog of which is the conservation law for energy. From this relation it follows that for a given choice of C the possible values of η are restricted to those for which $\Phi(\eta)\geq C$. Considering an infinitely extended system, and disregarding, for physical reasons, those solutions for which $|\eta(x)|$ is not bounded, we are left with three types of stationary states:

(a) Constant solutions η_i corresponding to the extrema of $\Phi(\eta)$ $[C_i = \Phi_i = \Phi(\eta_i)]$:

$$[\partial\Phi(\eta)/\partial\eta]_{\eta_i} = 0$$

(b) (Spatially) periodic solutions.

(c) Solitons.*

For the potential $\Phi(\eta)$ shown in Fig. 12a, the solutions of type (b) $(\Phi_2 > C > \Phi_3)$ and (c) $(C = \Phi_3)$ are sketched in Figs. 12b and 12c.

We add the following remarks: with $\eta_{st}(x)$, both $\eta_{st}(x+a)$ and $\eta_{st}(-x+a)$ are also solutions of Eq. (B3) (mechanical analog: time translation, time reversal); the curvature at the extrema of the periodic solutions is given by $(1/\xi^2)\,\partial\Phi/\partial\eta$ at the turning points $\Phi(\eta)=C$. The periodic solution degenerates to a soliton when one of these curvatures becomes zero, which is the case for $C=\Phi_3$. The soliton degenerates further to a wall (Fig. 12d) when both curvatures are zero, which happens when accidentally $\Phi_1=\Phi_3=C$.

For the discussion of the local stability of a stationary state $\eta_{st}(x)$, one has to study the dynamical behavior of $\eta(x,t)$ in the vicinity of η_{st}. Linearizing Eq. (B2) around η_{st} leads to

$$\dot{u} = -(\partial^2\Phi/\partial\eta^2)_{\eta_{st}}u + \xi^2 u'' \tag{B5}$$

where we have put $u(x,t)=\eta(x,t)-\eta_{st}(x)$. In order to find the normal

*Here solitons refer just to localized objects.

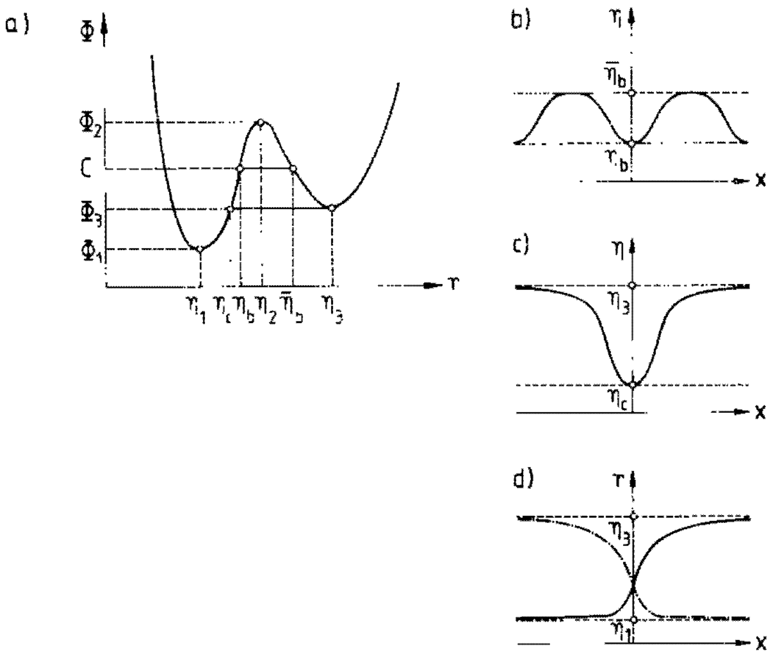


Fig. 12. Types of spatially inhomogeneous stationary states related to the potential Φ shown in (a). (b) Periodic solution ($\Phi_2 > C > \Phi_1, \Phi_3$). (c) Soliton ($C = \Phi_3$). (*d*) Soliton when accidentally $\Phi_1 = \Phi_3 = C$.

modes $u(x, t) = \exp(-\omega_n t)\, u_n(x)$, we have to solve a one-dimensional Schrödinger equation

$$-\xi^2 u_n'' + (\partial^2\Phi/\partial\eta^2)_{\eta_{st}} u_n = \omega_n u_n \tag{B6}$$

We discuss this problem for the three types of stationary states mentioned before:

(a) Constant stationary states η_i: In this case $(\partial^2\Phi/\partial\eta^2)_{\eta_i}$ is constant, too, and positive for a minimum of $\Phi(\eta)$. Hence, $\omega_n > 0$ for all n and therefore a minimum of $\Phi(\eta)$ is locally stable. On the other hand, a maximum of $\Phi(\eta)$ is necessarily locally unstable because there is a set of $\omega_n < 0$.

(b) Periodic stationary states $\eta_{st}(x)$: Differentiating both sides of Eq. (B3), we see that $u_0(x) = \eta_{st}'(x)$ is (the only) eigenfunction with an eigen value $\omega_0 = 0$. Because $u_0(x)$ has infinitely many nodes, there exist eigenfunctions with $\omega_n < 0$ and, hence, all periodic stationary states are locally unstable.

(c) Solitons: Because for a soliton of the type sketeched in Fig. 12c $u_0(x) = \eta_{st}'(x)$ has one node, there exists one eigenfunction $u_1(x)$ with $\omega_1 < 0$. For a soliton of the wall type (Fig. 12d) there exists no eigenfunction

$u_1(x)$ with $\omega_1 < 0$, because $u_0(x) = \eta'_{st}(x)$ has no node. Hence, a soliton of the type Fig. 12c is locally unstable, whereas a soliton of the wall type of Fig. 12d is marginally stable, which is related to the translational invariance of the system.

From the equation of motion (B2), we find that for a time-dependent solution $\eta(x, t)$

$$\begin{aligned}\dot{U}\{\eta\} &= \int dx \frac{\delta U\{\eta\}}{\delta\eta(x,t)} \dot{\eta}(x,t) \\ &= -\int dx \left(\frac{\delta U\{\eta\}}{\delta\eta(x,t)}\right)^2 \\ &< 0 \qquad \text{for} \quad \delta U/\delta\eta \neq 0 \\ &= 0 \qquad \text{for} \quad \delta U/\delta\eta = 0\end{aligned} \tag{B7}$$

Therefore $U\{\eta\}$ is a Liapunov function for the problem. From relation (B7) and Eq. (B2) we see that the system moves in the course of time toward a stationary state, where it will then stay forever. From the linear stability analysis we know that the minima of $\Phi(\eta)$ are the only stationary states that are stable against perturbations in arbitrary directions in the state space. The regions of attraction of all the other stationary states are of lower dimensionality. We therefore conclude that up to cases of measure zero an arbitrary state in the course of time will tend to a spatially homogeneous state which is a minimum of $\Phi(\eta)$. In particular, it follows that there exist no solutions which are period in time (limit cycles).

APPENDIX C

For spatially homogeneous states of a microwave-stimulated superconductor, the gap control is given by

$$\begin{aligned}\chi &= -B \int dE \frac{N_1(E)}{E} \left\{ N_1(E-\nu)\left[1 + \frac{\Delta^2}{E(E-\nu)}\right] - (\nu \to -\nu)\right\} \\ &= BG(\Delta/\nu)\end{aligned} \tag{C1}$$

The function G is found to be equal to

$$G(u) = \begin{cases} 2\pi u(1-u^2)^{1/2}, & u < \frac{1}{2} \\ [2/(u+\frac{1}{2})]\{K(k) + 4u^2[\Pi(\alpha^2, k) - K(k)]\}, & u > \frac{1}{2} \end{cases} \tag{C2}$$

In our notation of the elliptic integral we follow exactly the rules used in the book by Byrd and Friedman.[47] Furthermore, $k = (u - \frac{1}{2})/(u + \frac{1}{2})$ and $\alpha^2 = \frac{1}{4}(u + \frac{1}{2})^{-2}$.

The investigation of stability against spatially inhomogeneous fluctuations involves rather complicated expressions. In principle, one can avoid such forms which involve directly the Δ derivative of the density of states N_1. For instance, one can show that

$$z(u;x) = -\frac{\partial}{\partial u}\tilde{G}(u;x) + y(u;x)$$

$$\tilde{G}(u;x) = -\int dE \frac{N_1(E)}{E}\frac{1}{1+xN_1^{-1}}\phi_E \tag{C3}$$

$$y(u;x) = -2\frac{x}{u}\phi_\Delta + x^2\int dE \frac{\nu\Delta N_1}{E^3}\frac{1}{(1+xN_1^{-1})^2}\phi_E$$

where ϕ_E is the expression in the curly brackets of Eq. (C1).

The limit $x=\infty$ can be calculated by a rather direct method; one obtains

$$z(u;\infty) = \begin{cases} 2\pi(1-u^2)^{-1/2}, & u<\frac{1}{2} \\ (1/u)[G(u)-4K(k)+4(u+\frac{1}{2})E(k)], & u>\frac{1}{2} \end{cases} \tag{C4}$$

A graphical representation of this result is shown in Fig. 5a, which also contains a graph of the following function:

$$\left.\frac{\partial z(u;x)}{\partial x}\right|_{x=0} = -2\frac{\partial}{\partial u}J(u) + \frac{2}{u}\left[(1+2u)^{1/2} - \theta\left(\frac{1}{2}-u\right)(1-2u)^{1/2}\right] \tag{C5}$$

In the interval $0<u<\frac{1}{2}$, the function $J(u)$ is given by

$$J(u) = \ln\frac{(1+2u)^{1/2}+u+1}{(1-2u)^{1/2}-u+1} - u\frac{(1+2u)^{1/2}+(1-2u)^{1/2}}{1-u^2}$$
$$-\frac{1-2u^2}{(1-u^2)^{3/2}}\ln\frac{[(1-u^2)(1+2u)]^{1/2}+u+1-u^2}{[(1-u^2)(1-2u)]^{1/2}-u+1-u^2} \tag{C6}$$

Values of J in the range $u>\frac{1}{2}$ can be obtained, for instance, by analytical continuation and taking the real part of the above expression.

Finally, we remark that it is possible with a reasonable amount of labor to express $[\partial^2 z(u;x)/\partial x^2]_{x=0}$ and $z(u;1)$ in a closed form. In addition, we have performed some integrations numerically. Thus, we have obtained a consistent picture of the function $z(u;x)$.

APPENDIX D

The calculations in the case of phonon stimulation are very much the same as in the microwave case except for a change of sign in the coherence factors. If the gap control is $\chi = BG(\Delta/\nu)$, then

$$G = \begin{cases} -2\pi u(1-u^2)^{-1/2}, & u<\frac{1}{2} \\ [2/(u+\frac{1}{2})]\{K(k)-4u^2[\Pi(\alpha^2, k)-K(k)]\}, & u>\frac{1}{2} \end{cases} \tag{D1}$$

The elliptic integrals are defined as in Appendix C. It is obvious that the quantity $z(u;x)$ can be calculated by means of relations very similar to Eq. (C3). In addition, we list the following result:

$$z(u;\infty) = \begin{cases} -2\pi(1-u^2)^{1/2}, & u<\frac{1}{2} \\ \dfrac{1}{u}\left[G(u)-\dfrac{1}{u^2-\frac{1}{4}}K(k)-\dfrac{1}{u-\frac{1}{2}}E(k)\right], & u>\dfrac{1}{2} \end{cases} \tag{D2}$$

Instead of Eq. (C5), we have now

$$\frac{\partial z(u;x)}{\partial x} = -2\frac{\partial}{\partial u}J(u)+\frac{2}{u}\left[\frac{1}{(1+2u)^{1/2}}-\frac{\theta(\frac{1}{2}-u)}{(1-2u)^{1/2}}\right] \tag{D3}$$

where the function $J(u)$ is given by an expression very much the same as Eq. (C6) except for a change in sign of the second term and except for a factor $1-2u^2$ missing in the last term.

APPENDIX E

The gap control in the case of tunneling is given by $\chi = BG(\Delta/\Delta_G, |eV/\Delta_G|)$ where

$$G\left(\frac{\Delta}{\Delta_G}, \frac{|eV|}{\Delta_G}\right) = -\frac{eV}{\Delta_G}\int dE \frac{N_1(E)}{E} \times[N_1^G(E-eV)-N_1^G(E+eV)] \tag{E1}$$

Thus

$$G(u, v) = -2\theta(u+1-v)vg[(d-v)K(k)+(c-d)\Pi(\alpha^2, k)] \tag{E2}$$

where

$$g = \frac{2}{[(a-c)(b-a)]^{1/2}}; \qquad \alpha^2 = \frac{b-c}{b-d}; \qquad k^2 = \frac{\alpha^2(a-d)}{a-c} \tag{E3}$$

and a, b, c, and d are the parameters $v \pm 1$ and $\pm u$ assigned such that $a>b>c>d$.

APPENDIX F

It has been argued in the text that Eq. (66) can be replaced by Eq. (67) if $E \sim O(\Delta) \ll T_c$ and $\nu \ll T_c$. We discuss here the corrections that arise if the frequencies are allowed to be somewhat larger, say $\nu \lesssim T_c$. The results apply to the case of microwave and phonon radiation as well as to the case of quasiparticle injection, where the voltage eV replaces the frequency ν. Rather large phonon frequencies are of particular interest, since for this case a spatially modulated state is expected to be stable.

The arguments are very similar to those made in the derivation of EMS in Appendix A. Consider first the metal in the normal state. There, the drive term of Eq. (66) generates a correction n_0 to the distribution function which is, in its leading part, thermal. Strictly speaking, for the exact calculation of n_0 one needs the full electron–phonon collision operator, since n_0 extends to energies up to the order of T_c. However, for an order of magnitude calculation, we may employ the relaxation ansatz (7). With the same accuracy, we may replace the second difference of the Fermi functions of the drive term by the second derivative. Thus, we obtain

$$n_0 = \frac{\nu}{T_c} B \frac{sh(E/2T_c)}{ch^3(E/2T_c)} \tag{F1}$$

As before, one may convince oneself that the distribution function $n_T + n_0$ is a good solution for the superconducting state with accuracy Δ^2/T_c^2 if the coherence factors in the collision integral are neglected. Hence, we correct the distribution function by a term n_1 as shown in Eq. (A5), where the correction n_1 is obtained as the solution of the Boltzmann equation in which a combination of the remainders $\mp\Delta^2/EE'$ of the coherence factors appears as a drive term $(\dot{n}_p)_{D0}$. One recognizes that only the contribution of the form (A6) leads to an interesting control on the gap. Thus

$$(\dot{n}_p)_{D0} = \frac{\pi \ln 2}{7\zeta(3)} \frac{\nu}{\tau_E T_c} B \frac{\Delta^2}{\pi T_c E_p} \tag{F2}$$

where Eq. (A2) for $1/\tau_E$ has also been employed. The energy dependence of this drive term allows us to calculate n_1 in the relaxation approximation.

The contribution χ_0 of n_0 to the gap control is of the order $(\nu/T_c)B$ and independent of Δ with an accuracy of $O(\Delta^2/T_c^2)$. In particular, we obtain from Eq. (F1) the following relation:

$$\chi_0 = \frac{14\zeta(3)}{\pi^2} \frac{\nu}{T_c} B \tag{F3}$$

A numerical solution[48] for n_0 leads to an expression which is smaller by a factor 0.4 than the above result. A Δ-dependent contribution χ_1 to the gap control is obtained from n_1. Taking into account Eq. (F2), we find

$$\chi_1 = -\gamma(\nu/T_c)B\frac{\Delta}{T_c} \tag{F4}$$

where $\gamma \sim (\pi \ln 2)/7\zeta(3)$ is a constant of order unity. In case of tunnel injection, it follows from Eq. (81) that the quantity B above has to be replaced by $\Delta_G \tau_E / [16 T_c e^2 R \Omega N(0)]$ and ν by $|eV|$.

As an illustration, we mention a possible realization of NEMS. Consider a phonon radiation source with a linear spectral distribution up to a maximum frequency ν_0. Thus, we may put the number dN_R of quanta absorbed in the range ν to $\nu + d\nu$ equal to $2N_R^{(0)} \nu \, d\nu / \nu_0^2$. This necessitates an integration of the direct drive term (78) with respect to ν, and one obtains

$$(\dot{n}_p)_D = \frac{4B^{(0)}}{\tau_E} \frac{\Delta^2}{\nu_0 E} \tag{F5}$$

provided that $E \sim O(\Delta) \ll \nu_0$. As a consequence, the following direct contribution to the gap control arises:

$$\chi_d = -4\pi B^{(0)} \Delta / \nu_0 \tag{F6}$$

which is of the desired form. Opposed to this control is the indirect contribution, which results from an integration of Eq. (F4),

$$\chi_{\text{ind}} = \tfrac{2}{3} \gamma (\nu_0^2 / T_c^2) B^{(0)} \Delta / \nu_0 \tag{F7}$$

One expects that the direct contribution dominates even for $\nu_0 \sim T_c$.

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